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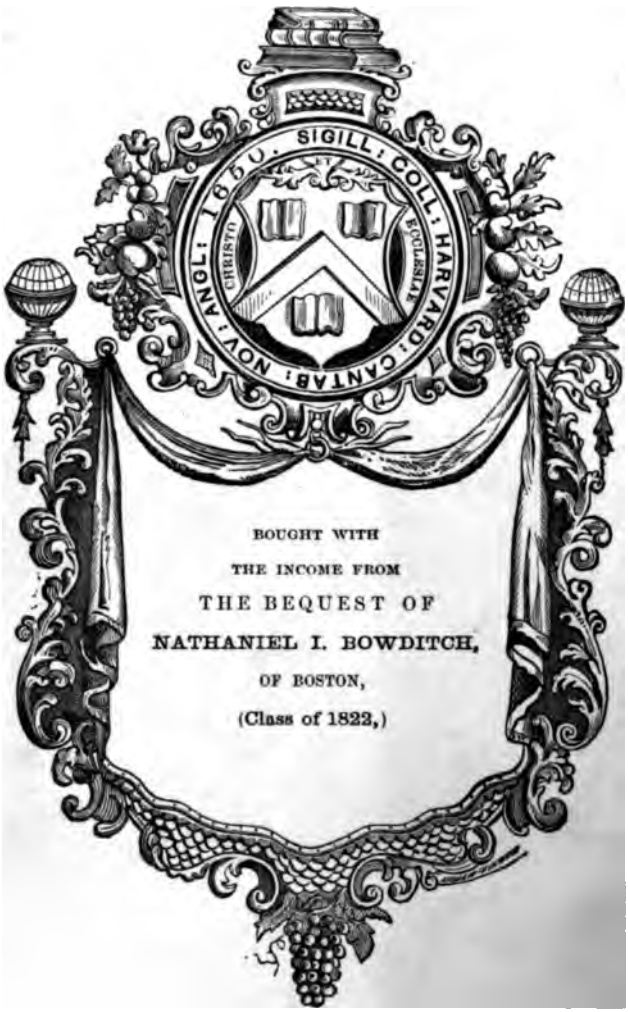

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MIRRORS, PRISMS AND LENSES

A TEXT-BOOK OF GEOMETRICAL OPTICS

BY

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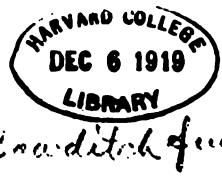

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PREFACE

In spite of the existence of a number of excellent works on geometrical optics, the need of a text-book which will serve as an introduction to the theory of modern optical instruments appears to be generally recognized; and the present volume, which is the outgrowth of a course of lectures on optics given in Columbia University, has been written in the hope that it may answer this purpose. In a certain sense it may be considered as an abridgment of my treatise on *The Principles and Methods of Geometrical Optics*, but the reader will also find here a considerable mass of more or less new and original material which is not contained in the larger book. I have endeavored, however, to keep steadily in mind the limitations of the class of students for whom the work is primarily intended and to employ, therefore, only the simplest mathematical processes as far as possible. With this object in view I have purposely entered into much detail in the earlier and more elementary portions of the subject, following in fact the method which has been found to be most satisfactory with my own pupils; but I venture to hope that the book may be not without interest also to readers who already possess a certain knowledge of the subject.

Recent years have witnessed extraordinary progress in both ophthalmology and applied optics. Not many persons are aware of the rapid rate at which spectacle optics, in particular, is developing into a severe scientific pursuit; and there are certain portions of this volume which I think will be helpful to the modern oculist and optometrist. Thus, for example, I have been at some pains to explain the fundamental principles of ophthalmic lenses and prisms.

In general, however, I have necessarily had to omit much

that is essential to a thorough knowledge of the theory of optical instruments. In fact, in the space at my disposal it has been found quite impossible to describe a single one of these instruments in detail. In the latter portion of the book the theory of the chromatic and spherical aberrations is treated as briefly as possible; and I have given Von Seidel's formulæ for the five spherical aberrations in the case of a system of infinitely thin lenses, chiefly because these formulæ are exceedingly useful in the preliminary design of an optical system. But a complete discussion of these subjects would lie far beyond the plan of this volume.

The problems appended to each chapter were originally collected for the use of my pupils and are generally of a very elementary description. A few of them have been adapted from other text-books, but in such cases I have now lost sight of their sources.

If perchance this book should help to stimulate the study of optics in our colleges and universities, the author will feel abundantly repaid. Unfortunately, at present geometrical optics would seem to be a kind of Cinderella in the curriculum of physics, regarded perhaps with a certain friendly toleration as a mathematical discipline not without value, but hardly permitted to take rank on equal terms with her sister branches of physics. On the contrary, it might be inferred that any system of knowledge which had already placed at the disposal of scientific investigators such incomparable means of research as are provided by modern optical instruments, and which has found so many useful applications in the arts of both peace and war, would be deserving of the highest recognition and would be fostered and encouraged in all possible ways. According to the maxim, *fas est et ab hoste doceri*, the fact that from the time of Fraunhofer the Germans have not ceased to cultivate this field of theoretical and applied science with notable achievements, is certainly not without significance for us in this country and in England. Indeed, both in England and in France,

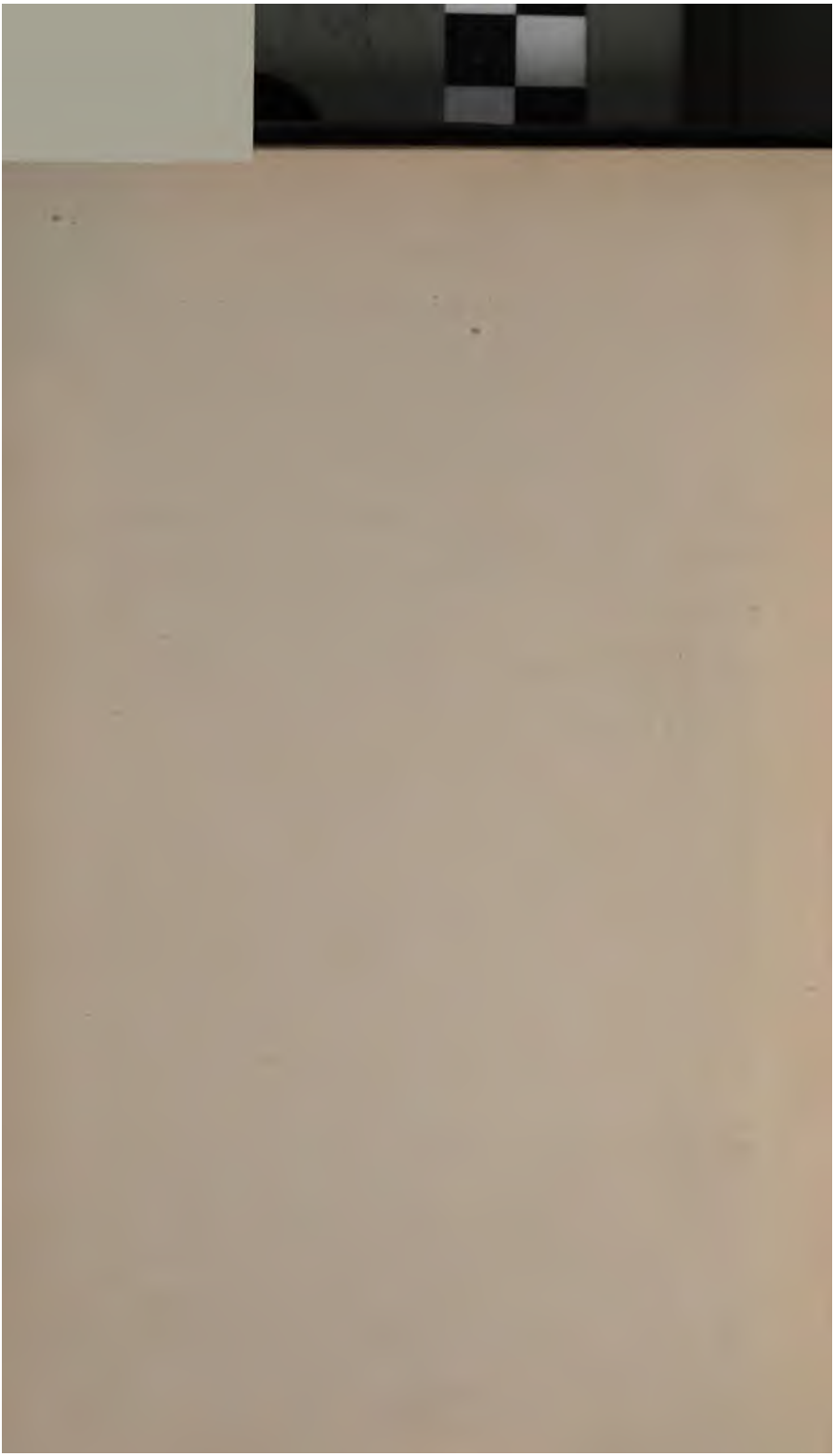
apparently due to the exigencies of war, schools of applied optics have recently been organized.

Nearly all of the diagrams in this volume were drawn by my friends, Professor Joseph Hudnut, Dr. B. A. Wooten and Mr. J. G. Sparkes, to whom I am much indebted. I desire also to express my grateful acknowledgments to my colleague, Professor H. W. Farwell, for numerous valuable criticisms from time to time and especially for aid in making the photographic illustrations in Chapter II.

Any suggestions or corrections which may improve and extend the usefulness of the book will be appreciated.

JAMES P. C. SOUTHALL.

Columbia University,
New York, N. Y.,
April 4, 1918.



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MIRRORS, PRISMS AND LENSES

MIRRORS, PRISMS AND LENSES

CHAPTER I

LIGHTS AND SHADOWS

1. Luminous Bodies.—The external world is revealed to the eye by means of light. With the rising sun night is changed into day, and animals, vegetables and minerals in all their manifold varieties of form and shade and color, which were quite invisible in the dark, are now revealed to view. Wherever the eye turns to gaze, there comes to it from far or near a messenger of light conveying information about the object which is under inspection. In an absolutely dark room everything is invisible, because the eye can perceive objects only when they radiate or reflect light into it. In the strict sense a source of light is a self-luminous body which shines by its own light, such as the sun or a fixed star or a candle-flame; but frequently the term is applied to a body which merely reflects or transmits light which has fallen upon it from some other body, as, for example, the moon and the planets which are illuminated by the light from the sun. In this latter sense the blue sky and the clouds, which, shining by light derived originally from the sun, contribute the greater portion of what is meant by daylight, are to be regarded as light-sources. A point-source of light or a luminous point is in reality a small element of luminous surface of relatively negligible dimensions or else a body like a star at such a vast distance that it appears like a point.

2. Transparent and Opaque Bodies.—In general, when light falls on a body, it is partly turned back or reflected at or very near the surface of the body, partly absorbed within

the body, and partly transmitted through it. An absolutely black body which absorbs all the light that falls on it does not exist; the best example we have is afforded by a body whose surface is coated with lamp-black. The color of a body as seen by reflected light is explained by the fact that part of the incident light is absorbed, whereas only light characteristic of the color in question is cast off or reflected from the body. Thus, when sunlight falls on a piece of red flannel, it is robbed of all its constituent colors except red, and thus it happens that the color by which we describe the body is in fact due to the light which it rejects. If the piece of red flannel were illuminated by pure blue light, it would appear black or invisible.

A substance such as air or water or glass, which is pervious to light, is said to be *transparent*. None of the light that traverses a perfectly transparent body will be absorbed; and, on the other hand, a perfectly *opaque* body is one which suffers no light at all to be transmitted through it. No substance is either absolutely transparent or absolutely opaque. These terms, therefore, as applied to actual bodies are merely relative, and so when we say that a body is opaque, we mean only that the light transmitted through it is so slight as to be practically inappreciable. Naturally, one thinks of clear water as transparent and of metallic substances generally as opaque; but a sufficiently large mass of water will be found to be impervious to light, whereas, on the other hand, gold leaf transmits green light. A perfectly transparent body would be quite invisible by transmitted light, although its presence could be detected by observing the distortion in the appearance of bodies viewed through it.

Again there are some substances which, while they are not transparent in the ordinary sense, are far from being opaque, such, for example, as ground glass, alabaster, porcelain, milk, blood, smoke, which contain imbedded or suspended in them fine particles of matter of a different optical

quality from that of the surrounding mass. Light does penetrate through materials of this nature in a more or less irregular fashion, and accordingly they are described as *translucent*. In the interior of such granular structures or "cloudy media" light undergoes a so-called internal diffused reflection or scattering; so that while it may be possible to discern the presence of a body through an intervening mass of such material, the form of the object will be to some extent indistinct and unrecognizable.

An *optical medium* is any space, whether filled or not with ponderable matter, which is pervious to light. In geometrical optics it is generally assumed that the media are not only homogeneous and isotropic (meaning thereby that the substance possesses the same properties in all directions), as, for example, air, glass, water and vacuum, but perfectly transparent as well.

3. Rectilinear Propagation of Light.—When an opaque body is interposed between the observer's eye and a source of light, it is well known that all parts of the latter which lie on straight lines connecting the pupil of the eye with points of the opaque obstacle will be hid from view. We cannot see round a corner; we can look through a straight tube but not through a crooked one. A child takes note of such facts as these among the very earliest of his experiences and recognizes without difficulty the truth of the common saying that "light travels in straight lines," which in the language of science is called the law of the rectilinear propagation of light. The light that comes to us from a star traverses the vast stretches of interstellar space in straight lines until it reaches the earth's atmosphere, which is composed of layers of air of increasing density from the upper portions towards the surface of the earth; so that the medium through which the light passes in this short remainder of its downward journey is no longer isotropic, and, hence, also this part of the light path will, in general, be no longer straight but curved by a gradual and continuous bending

from the less dense layers of air to the more dense layers below. This explains why it is necessary for an observer on the earth's surface looking through a long narrow tube at a star not directly overhead to point the tube not at the star itself but at its apparent place in the sky, which depends on the direction which the light has when it enters the eye; and, consequently, in accurate determinations of the position of a heavenly body, the astronomer is always careful to take account of the apparent displacement due to this so-called "atmospheric refraction," and a principal reason why astronomical observatories are nearly always located on high mountains is to obviate as much as possible the disturbing influence of the atmosphere. In aiming a rifle or in any of the ordinary processes we call "sighting," which are at the basis of some of the most delicate methods of measurement known to us, we rely with absolute confidence on this proved law of experience concerning the rectilinear propagation of light; and, in fact, the most conclusive demonstration that a line is straight consists in showing that it is the path which light pursues. The notion of a "ray of light" is derived from this law, and any line along which light travels is to be regarded as a ray of light. According to this idea, therefore, *the rays of light in an isotropic medium are straight lines.*

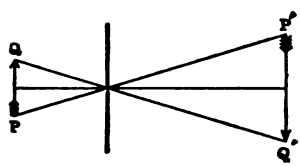


FIG. 1.—Rectilinear Propagation of Light.

A very striking proof of the rectilinear propagation of light is afforded by placing a luminous object (Fig. 1) in front of an opaque screen in which there is a very small round aperture. If now a second screen or a white wall is placed parallel to the first screen on the other side of it, there will be cast on it a so-called inverted image of the object, the size of which will be proportional to the distance between the two screens. From each point of the luminous object rays go out in all directions, and a narrow

cone of these rays will traverse the perforated screen through the opening and illuminate a small area on the other screen, and thus every part of the object will be depicted in this way by little patches of light arranged in a figure which is similar in form to the object, but which is completely inverted, since not only top and bottom but right and left are reversed in consequence of the rectilinear paths of the rays of light. It may be remarked that this image is not an optical image in the strict sense of the term (see § 11), but the phenomenon can be explained only on the supposition that light proceeds in straight lines. If another small opening were made in the front screen very near the first hole, there would be two images formed which would partly overlap each other, so that the resultant image would be more or less blurred, and if we have a single large aperture, we could no longer see any distinct image at all.

The *pinhole camera*, invented by GIAMBATTISTA DELLA PORTA (*c.* 1543–1615), and sometimes called PORTA'S camera, is constructed on the principle of the experiment which has just been described. It is very useful in making accurate photographic copies of the architectural details of buildings, because the image which is obtained is entirely free from distortion.

In the pinhole camera there is a certain relation between the size of the pinhole and the distance of the sensitive plate. According to ABNEY, in order to get the best results with an apparatus of this kind the diameter of the pinhole ought to be directly proportional to the square-root of the distance of the plate from the aperture, that is,

$$y = k\sqrt{x},$$

where x and y denote the distance of the plate and the diameter of the pinhole, respectively, and k denotes a constant, the value of which will depend on the unit of length. Thus, if x and y are measured in inches, $k = 0.008$; in centimeters, $k = 0.01275$.

4. Shadows, Eclipses, etc.—The forms of shadows are also easily explained on the hypothesis that light proceeds in straight lines, for the outline of the shadow cast by a body is precisely similar to that of the object as viewed from the place where the source of light is. Thus, for example, the

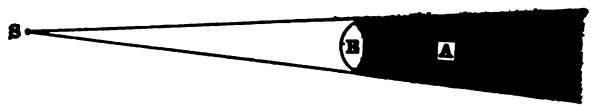


FIG. 2.—Shadow (umbra) of opaque globe E illuminated by point-source S.

shadow of a sphere held in front of a point-source of light has the form of a circle, and the shadow cast by a circular disk will have the outline of an ellipse of greater and greater eccentricity as the disk is turned more and more nearly edge-on towards the light. Passing a shop-window on Sunday when the shade is drawn down, if the sun is shining

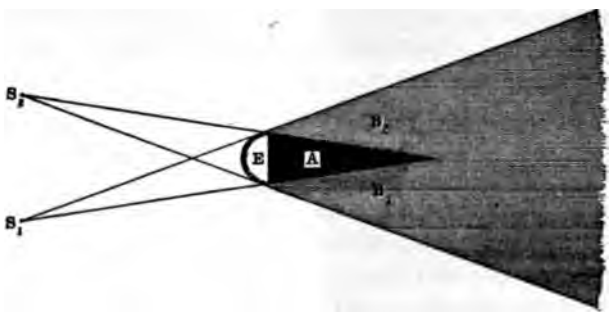


FIG. 3.—Shadow (umbra and penumbra) of opaque globe E illuminated by two point-sources S_1, S_2.

on the window, one can read the shadow of the sign painted on the glass quite as distinctly as the sign itself. The interposition of an opaque body between a source of light and a wall not only darkens a portion of the wall or casts its shadow there, but it converts an entire region of space between it and the wall into a dark tract either wholly or par-

tially screened from the light. Thus, for example, the space A (Fig. 2) behind the body E which is comprised within the cone of rays proceeding from the point-source S that are intercepted by E gets no light from S, and this wholly unilluminated region is called the *umbra* or true shadow. When there are two luminous points S_1 and S_2 (Figs. 3 and 4), the region of shadow behind the opaque body E consists of the

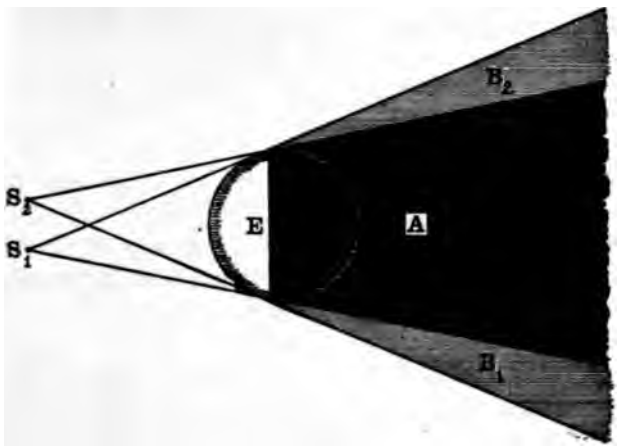


FIG. 4.—Shadow (umbra and penumbra) of opaque globe E illuminated by two point-sources S_1, S_2.

umbra A which is wholly screened from both sources of light and the so-called *penumbra* or partially illuminated space composed of a space B_1 which gets light only from S_1 and a similar space B_2 which gets light only from S_2. Points lying beyond the *penumbra* will receive light from both sources.

If the light-source has an appreciable size, light will proceed from each of its shining points in all directions. Suppose, for example, that an opaque globe E (Fig. 5) is placed in front of a luminous globe S: then the dark body will intercept all rays that fall within the cone which is tangent externally to the two spheres, and, consequently, the portion A of this cone which lies behind E will be completely

screened from all points of the source S, so that this portion constitutes the *umbra* where no light comes. In this case also there are two penumbral regions B_1 and B_2 which are partially illuminated, but the illumination is not uniform,

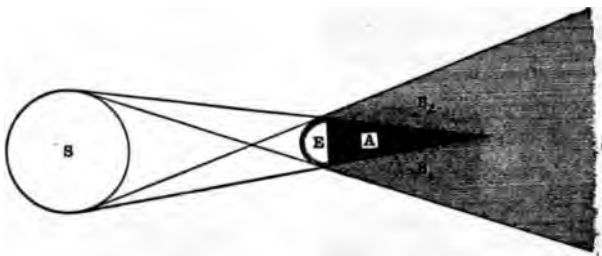


FIG. 5.—Shadow (umbra and penumbra) of opaque globe E illuminated by luminous globe S.

but increases gradually from total darkness at the outer borders of the *umbra* into the complete illumination of the region outside the shadow. The shadow cast on a screen by an opaque body exposed to an extended source of light has no sharp outline but fades by imperceptible gradations into the bright space outside. As to the *umbra*, it terminates in a point at a certain distance x behind the opaque body, provided the diameter of the latter is less than that of the luminous globe in front of it, that is, provided R is greater than r, where R, r denote the radii of luminous and opaque globes, respectively. If the distance d between the centers of the two globes is known, the length x of the *umbra* may be calculated from the proportion:

$$\frac{R}{r}=\frac{d+x}{x};$$

whence we find:

$$x=\frac{d}{\frac{R}{r}-1}.$$

Thus, for example, the diameter of the sun is 109.5 times that of the earth, and the distance between the two bodies

is 93 millions of miles. Accordingly, the *umbra* of the earth is found to extend to a distance of more than 857 000 miles behind it. Sometimes the moon whose distance from the earth is about 240 000 miles enters inside the shadow, and becomes then totally eclipsed. When the moon is only partly inside the earth's *umbra*, there is a partial eclipse of the moon. On the other hand, if the earth or any part of it comes inside the moon's shadow, there will be an eclipse of the sun visible from points on the earth that are in the shadow.

The angular diameter of the sun is 32′ 3.3″; whence it is easy to calculate that the length of the *umbra* of an opaque globe in sunlight is about 105 times the diameter of the globe.

On the other hand, if the light-source is smaller than the interposed object, the *umbra*, instead of contracting to a point, widens out indefinitely; and thus, whereas the shadow cast on the opposite wall by a hand held in front of a broad fire is smaller than the object, the shadow made by the same hand in front of a small source of light like a candle-flame may be prodigious in extent.

5. Wave Theory of Light.—The term "ray," as we have employed it, is a purely geometrical conception, but in ordinary usage a ray of light implies generally an exceedingly narrow beam of light such as is supposed to be obtained when sunlight is admitted into a dark room through a pin-hole opening in a shutter. But when the experiment is carefully made to try to isolate a so-called ray of light in this fashion, new and unexpected difficulties arise, and, contrary to our preconceived notions, we are disconcerted by finding that the smaller the opening in the shutter, the more difficult it becomes to realize the geometrical conception which is conveyed by the word "ray." In fact, in consequence of this experiment and others of a similar kind, we begin to perceive that the statement of the law of the rectilinear propagation of light needs to be modified; for among other phenomena we discover that when light pro-

ceeds through a very narrow aperture in a screen, it does not pass through it just as though the screen were not present, but it spreads out laterally from the point of perforation in all directions beyond the screen, proceeding, in fact, very much as it might do if the opening in the screen were the seat of a new and independent source of light.

The truth is, as has been ascertained now for a long time, light is propagated not by "rays" at all but by waves; and if, in general, it is found that light does proceed in straight lines and does not bend around corners as sound-waves do, the explanation is because the waves of light are excessively short, considerably less than one ten-thousandth of a centimeter. Wave-lengths of light are usually specified in terms of a unit called a "tenth-meter" or an "ÅNGSTROM unit," which is the hundred-millionth part of a centimeter (see § 162); that is, 1 ÅNGSTROM unit $= 10^{-10}$ meter $= 0.000\ 000\ 01$ cm. The wave-length of the deepest red light is found to be about 7667 of these units and the wave-length of light corresponding to the extreme violet end of the spectrum is a little more than half the above value or 3970 units.

According to the wave-theory the phenomena of light are dependent on an hypothetical medium called the *ether*, which may be compared to "an impalpable and all-pervading jelly" that not only fills empty space but penetrates freely through all material substances, solid, liquid and gaseous, and through which particles of ordinary matter move easily without apparent resistance, for it is imponderable and exceedingly elastic and subtle, insomuch that no one has ever succeeded in obtaining direct evidence of its existence. It is this ether which is the vehicle by which light-energy is transmitted and through which waves of light are incessantly throbbing with prodigious but measurable velocity, which *in vacuo* is about 300 million meters per second or about 186 000 miles per second.

6. Huygens's Construction of the Wave-Front.—The great Dutch philosopher HUYGENS (1629–1695), who was a contem-

porary of NEWTON'S (1642–1727), and who is usually regarded as the founder of the wave-theory of light, encountered his greatest difficulty in trying to give a consistent and satisfactory explanation of the apparent rectilinear propagation of light. His mode of reasoning, as set forth in his "Treatise on Light" published in 1690, while by no means free from objection, leads to a simple *geometrical construction of the wave-front* which corresponds with the known facts in regard to the procedure of light.

Let O (Fig. 6) designate the position of a point-source of light from which as center or origin ether waves proceed in an isotropic medium with equal speeds in all directions. At the end of a certain time the disturbances will have arrived at all the points which lie on a spherical surface σ_1 described around O as center, and at the instant in question this surface will be the locus of all the particles in the medium that are in this initial phase of excitation, and so it represents the *wave-front* at this moment. Now according to HUYGENS, every point in the wave-front becomes immediately a new source or center from which so-called secondary waves or wavelets spread out. These innumerable ripples or wavelets starting together from all the points affected by the principal wave overlap and interfere with each other, and HUYGENS inferred that their resultant sensible effects are produced only at the points of the surface which at any given instant touches or, as we say, envelops all the secondary wave-fronts, and that accordingly the new principal wave-

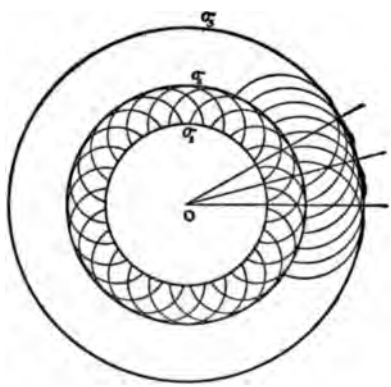


FIG. 6.—HUYGENS's construction of wave-front.

front will be this enveloping surface; so that the effect is the same as though the old wave-front had expanded into the new, the disturbance marching forward along a straight line in any given direction. Obviously, in an unobstructed isotropic medium, such as is here supposed, the enveloping surface or new wave-front will be a sphere concentric with the old wave-front, and the straight lines that radiate out from the center will be the paths of the disturbance.

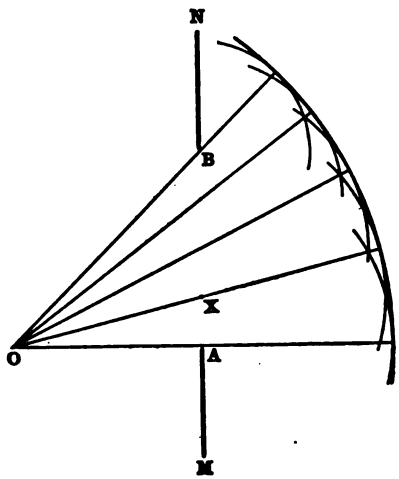


FIG. 7.—HUYGENS's construction of spherical waves passing through opening in a screen.

Now if a plane screen MN (Fig. 7) is interposed in front of the advancing waves, and if there is an opening AB in the screen, each point in the opening between A, which is nearest to the source O, and B, which is farthest from it, will become in turn a new center of disturbance whence secondary spherical waves will be propagated into the region on the other side of the screen. Since the disturbance will have arrived at the point A before it has reached a point X between A and B, the secondary wave emanating from A will at the end of a given time t have been travelling for a longer time than the secondary wave coming from X. If the radius of the wavelet around X at the time t is denoted by r, and if the distance OX is put equal to x, then $d=x+r$ will denote the distance from O which the disturbance will have gone at the end of the time t; and since this distance is constant, whereas the distances denoted by x and r are

variables depending on the position of the point X, it is evident that the farther X is from O, that is, the greater the value of x, the smaller will be the radius $r = d - x$ of the secondary wavelet around X. The enveloping surface in this case is seen to be that part of the spherical surface described around O as center with radius equal to d which is intercepted by the cone which has O for its vertex and the opening AB in the screen for a section. Within this cone, according to HUYGENS'S view, the disturbance is propagated exactly as though the perforated screen had not been interposed, whereas points on the far side of the screen and outside this limiting cone are not affected at all. It is plain that this mode of explanation is equivalent to the hypothesis of the rectilinear propagation of light.

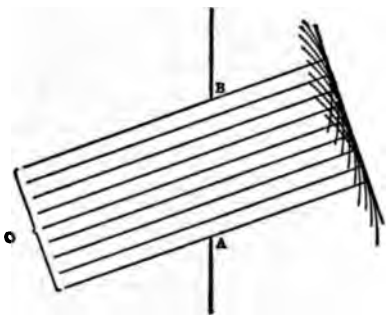


FIG. 8.—HUYGENS'S construction of plane waves passing through opening in a screen.

If the luminous point O (Fig. 8) is so far away that the dimensions of the opening AB in the screen may be regarded as vanishingly small in comparison with the distance of the source, the straight lines drawn from O to the points A, X, B in the opening in the screen may be regarded as parallel, and the wave-front in this case will be plane instead of spherical, that is, the wave-front is a spherical surface with an exceedingly great radius as compared with the dimensions of the aperture in the screen.

7. Rays of Light are Normal to the Wave-Surface.—The most obvious objection to HUYGEN'S construction is, What right has he to assume that the places of sensible effects are the points on the surface which is tangent to or envelops the secondary waves? And why is the light not propagated

backwards from these new centers as well as forwards? Moreover, when the opening in the screen is very narrow, it is found, as has been already stated (§ 5), that this construction does not correspond at all with the observed facts.

It is entirely beyond the scope of this book to attempt to answer these questions here or to describe even briefly the remarkable and complex phenomena of *diffraction* (which is the name given to these effects due to the bending of the light-waves around the edges of opaque obstacles). For an adequate discussion of these matters the reader must consult a more advanced treatise on physical optics. Suffice it to say, that the wave-theory of light and especially the principle of interference as developed long after HUYGENS'S death (1695) by YOUNG (1773–1829) and FRESNEL (1788–1827) entirely supports the idea of the rectilinear propagation of light as commonly understood; notwithstanding the fact that this law, as indeed is the case with nearly all so-called natural laws, has to be accepted with certain reservations; but, fortunately, these latter do not concern us at present.

Accordingly, a luminous point is said to emit light in all directions, and the so-called light-rays in an isotropic medium are straight lines radiating from the center of the spherical wave-surface. These rays may subsequently be bent abruptly into new directions in traversing the boundary between one isotropic medium and another, and under such circumstances the wave-surfaces may cease to be spherical; but no matter what may be the form of the wave-surface, *the direction of the ray at any point is to be considered always as normal to the wave-front that passes through that point* (see § 39). In an isotropic medium the waves always march at right angles to their own front, and the so-called rays of light in geometrical optics are, in fact, the shortest optical routes along which the disturbances in the ether are propagated from place to place. With the aid of the principle of interference (alluded to above) and by the use of the higher

mathematics, it may indeed be shown that the effect produced at any point P in the path of a ray of light is due almost exclusively to previous disturbances which have occurred successively at all the points along the ray which lie between the source and the point P in question, and that disturbances at other points not lying on the ray which goes through P are practically without influence at P, that is, their effects there are mutually counteracted. And thus we arrive also at the so-called *principle of the mutual independence of rays of light*, which is also one of the fundamental laws of geometrical optics. From this point of view a ray of light is to be regarded as something more than a mere geometrical fiction and as having in some real sense a certain physical existence, although it is not possible to isolate the ray from its companions.

8. The Direction and Location of a Luminous Point.— When a ray of light comes into the eye, the natural inference as to its origin is that the source lies in the direction from which the ray proceeded. There is no difficulty in pointing out correctly the direction of an object which is viewed through an isotropic medium; but if the medium were not isotropic, the apparent direction of the object might not be, and probably would not be, its real direction. Thus, owing to the effects of atmospheric refraction, to which allusion has been made already (§ 3), the sun is seen above the horizon before it has actually risen, and so also in the evening the sun is still visible for a few moments after sunset. For the same reason a star appears to be nearer the zenith than it really is.

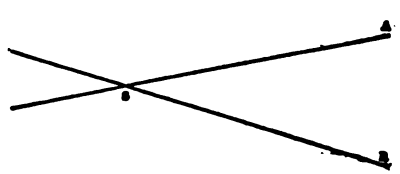


FIG. 9.—Direction and location of a luminous point.

In general, however, when a ray SA (Fig. 9) enters the eye at A, it is correctly inferred that the source S lies somewhere on the straight line AS, but whether it is actually

situated at S or farther or nearer cannot be determined by means of a single ray. If the eye is transferred from A to another point B, the source will appear now to lie in the new direction BS. If the spectator views the source with both eyes simultaneously, one eye at A and the other at B, or if using only one eye he moves it quickly from A to B, the position of the source at S will be located at the point of intersection of the straight lines AS and BS; and this determination will be more accurate in proportion as the distance between the two points of observation A and B is greater or the more nearly the acute angle ASB approaches a right angle. That is the reason why in estimating the distance of a remote object one tries to observe it from two stations as widely separated as possible, and that explains also why a person shifts his head from side to side. If the object is comparatively near at hand, a single movement of the head may be sufficient in order to get a fairly good idea of its distance, or it may be that it is simply necessary to look at the object with both eyes at the same time. It is amusing to watch a person with one eye closed attempting to poke a pencil through a finger-ring suspended in the middle of a room on a level with his eye; by chance he may succeed after repeated failures, whereas with both eyes open, the operation is performed without the slightest difficulty.

In case the rays come into the eye after having traversed two or more isotropic media, it is easy to be deceived about the direction of the source where they emanated. In order for a bullet to hit a fish under water, the rifle must be pointed in a direction below that in which the fish appears to be. At the boundary-surface between two isotropic media the direction of a ray of light is usually changed abruptly by refraction (§ 26); so that, in general, the path of a ray will be found to consist of a series of line-segments. In Fig. 10 the broken line ABCD represents the course taken by a ray of light in proceeding through several media such as water, air and glass. The line-segments AB, BC and CD

are portions of different straight lines of indefinite extent. For example, the actual route of the ray in air is along the straight line between B and C, and if the point P lies on this line between B and C, we say that the ray BC passes "really" through P, whereas we say that this same ray passes "virtually" through a point Q or R which lies in the prolongation of the line-segment BC in either direction.

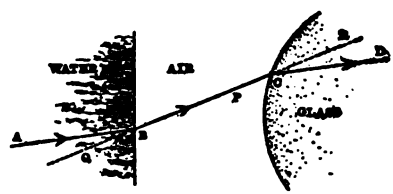


FIG. 10.—Points P, Q and R considered as lying on ray BC are to be regarded as lying in same medium as BC.

Moreover, thinking of the point Q or R as a point lying on the straight line BC which the light pursues in traversing the medium between the water and the glass, we must regard such a point as being optically in the same medium as the ray to which it belongs. Thus, the points Q and R in the figure considered as points on the ray BC are to be regarded as being optically in air, although in a physical sense Q is a point in the water and R is a point in the glass (see § 104).

Now let us suppose that two rays emanating originally from a point-source S (Figs. 11 and 12) are bent at A and B into new directions AP and BQ, respectively, so as to enter the two eyes of an observer at P and Q. In such a case the observer will infer that the rays originated at the point S′ where the straight lines AP and BQ intersect. This point S′, which is called the *image* of S, may lie in the actual paths of the rays AP and BQ that enter the eyes, so that the light from S really does go through S′, and in this case (Fig. 11)

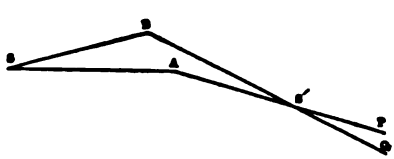


FIG. 11.—S′ is said to be a "real" image of point-source at S.

the image S′ is said to be a *real image*. On the other hand, if the straight lines AP and BQ have to be produced backwards in order to find their point of intersection, the rays do not actually pass through S′, and in this case the image is said to be a *virtual image* of the point S (Fig. 12). However, it must be borne in mind in connection with these diagrams that in reality we do not see objects by means of single rays; and, hence, we shall not be in a position to form an accurate idea of the term optical image until we come to consider bundles of rays in § 11.

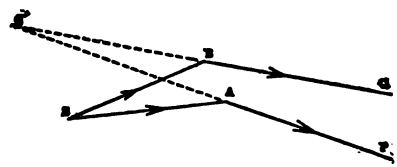


FIG. 12.—S′ is said to be a "virtual" image of point-source at S.

9. Field of View.—The open or visible space commanded by the eye is called the field of view. Since the eye can turn in its socket, the field of view of the mobile eye is very much more extensive than that of the stationary eye, and, moreover, the field of view of both eyes is greater than that of one eye by itself. The spectator may also widen his field of vision by turning his head or indeed by turning his entire body. For the present, however, we shall employ the term field of view to mean that more limited portion of space which is accessible to the single eye turning in its socket around the so-called center of rotation of the eye. When a person gazes through a window, the outside field of view is limited partly by the size of the window and partly also by the position of the eye with reference to it; so that only such exterior objects will be visible as happen to lie within the conical region of space determined by drawing straight lines from the center of rotation of the eye to all the points in the edge of the window. Thus, for example, if the opening in the window is indicated by the gap GH in the straight line GH in Fig. 13, and if the point marked O is the position of the center of rotation of the eye, a luminous object at P

in front of the window and directly opposite the eye will be plainly in view, because some of the rays from P may go through the window and enter the eye. But if the object is displaced far enough to one side to some position such as

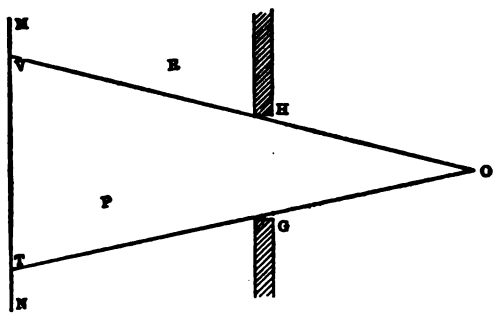


FIG. 13.—Field of view determined by contour of window GH and position of the eye at O.

that marked R in the diagram, so that the straight line OR does not pass through the window, the object will pass out of the field of view. The straight line MN drawn parallel to GH is supposed to represent a vertical wall opposite the window. If this wall is covered with a mural painting, the only part of the picture that can be seen through the window by the eye at O is the section included between the points T and V where the straight lines OG and OH intersect the straight line MN. The window acts here as a so-called *field-stop* (§ 137) to limit the extent of the field of view. But the limitation of the visible region depends essentially also on the position of the eye, becoming more and more contracted the farther the eye is from the window. The size of the window makes very little difference when the eye is placed close to it, and a person sitting near an open window can command almost as wide a view as if the entire wall of the room were removed. If one is looking through a keyhole in a door, he must put his eye close to the hole in order to see objects that are not directly in front of it.

10. Apparent Size.—The *apparent size* of an object is measured by the *visual angle* which it subtends at the eye. Several objects in the field of view which subtend equal angles when viewed from the same standpoint are said to have the same apparent size; although their actual sizes will

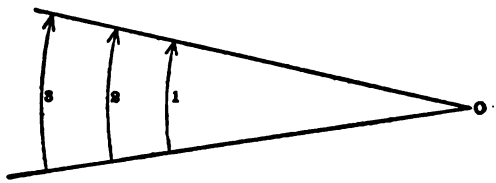


FIG. 14.—Apparent size measured by visual angle.

be different if they are at unequal distances from the eye. The objects marked 1, 2 and 3 in Fig. 14 appear to an eye at O to be all of the same size. Thus an elephant may appear no bigger than a man or a boy. Looking through a single pane of glass in a window, one may see a large building or an entire tree, because the apparent extent of the small area of glass is greater than that of the distant object. A fly crawling across the window may hide from view a large portion of the distant landscape outside. A mountain a few miles off may be viewed through a finger-ring.

The apparent size of an object, being measured by the visual angle which it subtends, is expressed in degrees or radians. The apparent diameter of the full moon in the sky, for example, is not quite half a degree, so that by holding a coin a little less than 9 mm. in diameter at a distance of one meter from the eye, the entire moon could be hid from view. In fact, instead of the angle itself it is customary to employ the tangent of the angle, especially in case the visual angle is not large. Thus, the apparent size of an object of height h at a distance d from the eye (in Fig. 15 AB = h, AO = d) is measured by the tangent of the angle BOA, that is,

$$\text{Apparent size} = \frac{\text{linear dimension of the object}}{\text{distance from the eye}} = \frac{h}{d}.$$

Accordingly, in order to determine the actual size (h) of the object, it is necessary to know its distance (d) as well as its apparent size, because the actual size is equal to the product of these two magnitudes. The apparent size of an

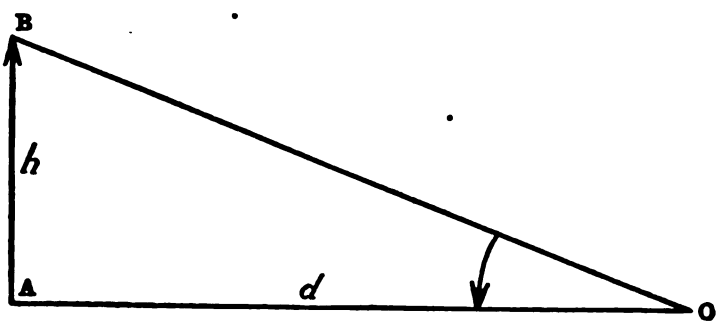


FIG. 15.—Apparent size varies inversely as distance d and directly as actual size h.

object at a distance of one foot is an hundred times greater than it is at a distance of an hundred feet, or, as we say, *the apparent size varies inversely as the distance.* As the object recedes farther and farther from the eye, its apparent size diminishes until at last it looks like a mere speck and the details in it have all disappeared. On the other hand, although the object is quite close to the eye, its actual dimensions may be so minute that it is not to be distinguished from a point. There is, indeed, a limit to the power of the human eye to see very small objects, which is reached when the object subtends in the field of view an angle that does not exceed one minute of arc. Two stars whose angular distance apart is less than this limiting value cannot be seen as separate and distinct by a normal eye without the aid of a telescope. Now $\tan 1' = \frac{1}{3438}$, and consequently the eye cannot distinguish details of form in an object which is viewed at a distance 3438 times as great as its greatest

linear dimension. A silver quarter of a dollar is about 24 mm. in diameter and viewed from a distance of 82.5 meters (3438 times 24 mm. = 82 512 mm. = 82.5 m.) its apparent size will be 1′ of arc and it will appear therefore like a mere point. The apparent width of a long straight street diminishes in proportion as the distance increases; until, finally, if the street is long enough, the two opposite sidewalks seem to run together at the so-called "vanishing point."

If rays of light coming through a window and entering the eye could leave marks in the glass at the points where they cross it, and if these marks could be made to emit the same kind of light as was sent out from the corresponding points of the object, there would be formed on the glass a pictorial representation of the object which when held before the eye at the proper distance would have almost exactly the same appearance as the object itself. This principle of *perspective* is made use of in the art of painting, and the artist, with his lights and shades and colors, tries to portray on a plane canvas a scene which will produce as nearly as possible the same visual impression on a spectator as would be produced by the natural objects themselves. So far as apparent size is concerned, such a representation may be perfect. In a good drawing the various figures are delineated in such dimensions that when viewed from the proper standpoint they have the same apparent sizes as the realities would have if seen under the aspect represented in the picture. No one looking at a photograph of a Greek temple will notice (unless his attention is specially directed to it) that the more distant pillars are much shorter in the picture than the nearer ones. Indeed, generally we pay little heed to the apparent sizes of things, but always try to conceive their real dimensions. When two persons meet and shake hands, neither is apt to observe that the other appears much taller than he did when they were fifty yards apart.

11. The Effective Rays.—All the rays that enter the eye and fall on the retina must pass through the circular window in the iris or colored diaphragm of the eye which is called the *pupil of the eye* and which is sometimes spoken of as the "black of the eye," because it appears black against the dark background of the posterior chamber of the eye. The pupil of the eye is about half a centimeter in diameter, although within certain limits its size can be altered to regulate the quantity of light which is admitted to the eye. So far as the spectator's vision is concerned, it is only these rays that go through the pupil of his eye that are of any use, and these are the *effective rays*. When the pupil dilates, more rays can enter, and consequently the source appears brighter. The brightness of the source will depend also on its distance, because for a given diameter of the pupil, the aperture of the cone of rays from a nearer source will be wider than that of the cone of rays from a more distant source. In general, therefore, the pupil of the eye regulates the angular apertures of the cones of rays that enter the eye from each point of a luminous object and acts as the so-called *aperture-stop* (§ 134). Thus, while the extent of the field of view is controlled by the field-stop (§ 9), the brightness of the source depends essentially on the size of the aperture-stop.

A series of transparent isotropic media each separated from the next by a smooth, polished surface constitutes an *optical system*. An *optical instrument* may consist of a single mirror, prisms or lens, but generally it is composed of a combination of such elements, which may be in contact with each other or separated by air or some other medium. In the great majority of actual constructions the instrument is symmetrical with respect to a straight line called the *optical axis*. Not all the rays emitted by a luminous object will be utilized by the instrument; generally, in fact, only a comparatively small portion of such rays will be transmitted through it, in the first place because its lateral di-

mensions are limited, and in the second place because, in addition to the lens-fastenings and other opaque obstacles (sides of the tube, etc.), nearly all optical instruments are provided with perforated screens or diaphragms called "stops," specially placed and designed to intercept such rays as for one reason or another it is not desirable to let pass (§133). The planes of these stops are placed at right angles to the optical axis with the centers of the openings on the axis. Accordingly, each separate point of the object is to be regarded as the vertex of a limited cone or *bundle of rays*, which, with respect to the instrument, are the so-called *effective rays*, because they are the only rays coming from the point in question that traverse the instrument from one end to the other without being intercepted on the way.

Moreover, in every bundle of rays there is always a certain central or representative ray, coinciding perhaps with the axis of the cone or distinguished in some special way, called the *chief ray* of the bundle (§ 139). In a symmetrical optical instrument the chief ray of a bundle of effective rays is generally defined to be that ray which in traversing a certain one of the series of media crosses the optical axis at a prescribed point, which is usually at the center of that one of the stops which is the most effective in intercepting the rays and which, therefore, is called the aperture-stop, as will be explained more fully hereafter (see Chapter XII). According to this definition, the chief rays coming from all the various points of the object constitute a bundle of rays which in the medium where the aperture-stop is placed (sometimes called the "stop medium") all pass through the center of the stop.

We shall employ the term *pencil of rays* to mean a section of a ray-bundle made by a plane containing the chief ray.

The effective rays in the first medium before entering the instrument are called the *incident rays* or *object rays;* and these same rays in the last medium on issuing from the instrument are called the *emergent rays* or *image rays*. If we

select at random any point X lying on one of the rays of the bundle of emergent rays which had its origin at the luminous object-point P, in general, no other ray of this bundle will pass through X, since in a given optical system there will usually be one single route by which light starting from the point P and traversing the instrument can arrive finally, either really or virtually (§ 8), at a selected point X in the last medium. However, there may be found a number of singular points where two or more rays of the bundle of emergent rays intersect; and under certain favorable and exceptional circumstances it may indeed happen that there is one special point P′ where *all* the emergent rays emanating originally from the object-point P meet again; and then we shall obtain at P′ a perfect or *ideal image* of P, which is described by saying that P′ *is the image-point conjugate to the object-point at* P. This image will be real or virtual according as the actual paths of the image-rays go through P′ or merely the backward prolongations of these paths (§ 8).

In order to obtain an image in this ideal sense, the optical system must be such as to transform a train of incident spherical waves spreading out from the object-point P into a train of emergent spherical waves converging to or diverging from a common center P′ in the image-space. When all the rays of a bundle meet in one point, the bundle of rays is said to be *homocentric* or *monocentric*. In general, however, a monocentric bundle of rays in the object-space will be transformed in the image-space into an *astigmatic bundle* of emergent rays, which no longer meet all in one point; and in fact this is a usual characteristic of a bundle of optical rays.

PROBLEMS

1. Why are the shadows much sharper in the case of an arc lamp without a surrounding globe than with one?

2. Draw a diagram to show how a total eclipse of the

moon occurs; and another diagram to illustrate a total eclipse of the sun. Give clear descriptions of the drawings.

3. An opaque globe, 1 foot in diameter, with its center at a point C, is interposed between an arc lamp S and a white wall which is perpendicular to the straight line SC. If the wall is 12 feet from the lamp, and if the distance SC=3 feet, what is the area of the shadow on the wall? Ans. 12.57 sq. ft.

4. What is the apparent angular elevation of the sun when a telegraph pole 15 feet high casts a shadow 20 feet long on a horizontal pavement? Ans. 36° 52′ 10″.

5. What is the height of a tower which casts a shadow 160 feet long when a vertical rod 3 feet high casts a shadow 4 feet long? Ans. 120 feet.

6. An object 6 inches high is placed in front of a pinhole camera at a distance of 6 feet from the aperture. What is the size of the inverted image on the ground glass screen if the length of the camera-box is 1 foot? Ans. 1 inch.

7. A small hole is made in the shutter of a dark room, and a screen is placed at a distance of 8 feet from the shutter. The image on the screen of a tree outside 120 feet away is measured and found to be 3 feet long. How high is the tree? Ans. 45 feet.

8. If the sensitive plate of a pinhole camera is 20 cm. from the pinhole, what should be the diameter of the pinhole, according to ABNEY's formula? Ans. 0.57 mm.

9. What is the apparent size of a man 6 feet tall at a distance of 100 yards? How far away must he be not to be distinguishable from a point? Ans. 1° 8′ 45″; 3.9 miles.

10. If the moon is 240 000 miles from the earth and its apparent diameter is 31′ 3″, what is its actual diameter? Ans. 2168 miles.

11. A person holding a tube 6 inches long and 1 inch in diameter in front of his eye and looking through it at a tree moves backwards away from the tree until the entire

tree is just visible. What is the apparent height of the tree? Ans. 9° 27′ 44″.

12. Assuming that the resolving power of the eye is one minute of arc, at what distance can a black circle 6 inches in diameter be seen on a white background? Ans. 1719 feet.

CHAPTER II

REFLECTION OF LIGHT. PLANE MIRRORS

12. Regular and Diffuse Reflection.—When a beam of sunlight, admitted through an opening in a shutter in a dark room, falls on a piece of smoothly polished glass, although the glass itself may be almost or wholly invisible, a brilliant patch of light will be reflected from the glass on the walls of the room or the ceiling or on some other adjacent object. If a person in the room happens to be looking towards the piece of glass along one special direction, he will be almost blinded by the light that is reflected into his eyes. The glass acts like a mirror and reflects the sunlight falling on it in a definite direction which depends only on the direction of the incident rays and on the orientation of the reflecting surface, and in such a case the light is said to be *regularly reflected*. Thus, for example, signals may be communicated to distant and inaccessible stations by reflecting thither the rays of the sun by a plane mirror adjusted in a suitable position.

If the surface is not smooth, the light will be reflected in many directions at the same time. The long sparkling trail of sunlight seen on the surface of a lake or a river on a bright day is caused by the reflections of the sun's rays into the eyes of the spectator from countless little ripples on the surface of the water.

The bright spot of light on the wall of a dark room at the place where a beam of sunlight falls, which shines almost as though this portion of the wall were itself a self-luminous body, is visible from any part of the room by means of the light which is reflected from it; and although the incident rays have a perfectly definite direction, the reflected light

is scattered in all directions. Some of this reflected light will fall on other bodies in the room, which will be more or less feebly illuminated thereby and rendered dimly visible by the light which they reflect in their turn; until at last the light after undergoing in this way repeated reflections from one body to another becomes too faint to be perceptible. Light which is reflected or scattered in this way is said to be *diffusely reflected* or *irregularly reflected*, although, strictly speaking, there is nothing irregular about it. Ordinarily it is in this way that bodies illuminated by daylight or by artificial light are rendered visible to a whole group of spectators at the same time.

The paper on the walls of an apartment which gets very little light through the windows should be a dull white in order to scatter and diffuse as much as possible the light that comes into the room. The walls of a dark chamber used for developing photographic plates should be painted a dull black in order to absorb the light that falls on them. An absolutely black body (§ 2) exposed to the direct rays of the sun will be completely invisible, except by contrast with its surroundings. If the walls of a dark room and all the objects within it were coated with lampblack, and if the air inside were entirely free from dust and moisture, a beam of sunlight traversing the room could not be seen and the only way to detect its presence would be by placing the eye squarely in its path. But if a little finely divided powder were scattered in the air or if a cloud of smoke were blown across the beam of light, the course of the rays would immediately become manifest to a spectator in any part of the room, because some of the light reflected from the floating particles of matter in practically every direction would enter the eye. But the light itself is quite invisible.

Any surface that is not too rough, that is, whose scratches or ridges are not wider than about a quarter of a wavelength of light, will reflect light in a greater or less degree depending on the smoothness of the surface. Waves of

light falling on a sheet of white paper are broken up or scattered in all directions, and we can get some idea of the quantity of light that is diffusely reflected from such a surface by letting the light of a lamp shine on the paper when it is held near an object that is in shadow. It is almost startling to see how under the influence of this indirect illumination the details of the obscure body suddenly appear as if summoned forth by magic. A highly polished metallic surface makes the best mirror, reflecting sometimes as much as three-fourths of the incident light. Our ordinary looking-glasses are really metallic mirrors, because they are coated at the back with silver, and the glass merely serves as a protection for the reflecting surface.

13. Law of Reflection.—A ray of light represented in Fig. 16 by the straight line AB is incident at B on a smooth reflecting surface whose trace in the plane of the diagram is the line ZZ. The straight line BN normal to the surface

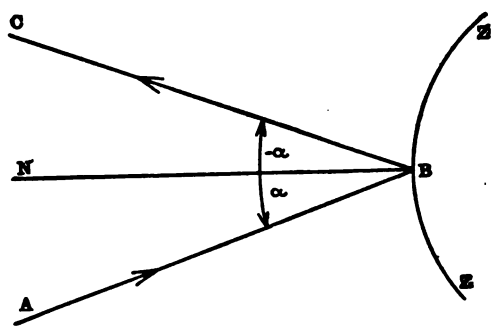


FIG. 16.—Law of reflection:
$\angle NBA = -\angle NBC = \angle CBN$.

at B is called the *incidence-normal*, and the plane ABN which contains the *incident ray* AB and the normal BN is called the *plane of incidence*, which corresponds here with the plane of the diagram. The *angle of incidence* is the angle between the incident ray and the incidence-normal; or, to

define this angle more exactly, *the angle of incidence is the acute angle* (α) *through which the incidence-normal has to be turned about the point of incidence in order to make it coincide with the incident ray;* thus, $\alpha = \angle$NBA. Counter-clockwise rotation is to be reckoned as positive and clockwise rotation as negative. This rule will be consistently observed in the case of all angular measurements.

The *reflected ray* corresponding to the incident ray AB is represented by the straight line BC; and if in the above definition of the angle of incidence we substitute "reflected ray" for "incident ray," we shall obtain the definition of the *angle of reflection* (β); that is, $\beta = \angle$NBC. The sense of the rotation is indicated by the order in which the letters specifying the angle are named; thus, $\angle$ABC is the angle described by rotating the straight line AB around the point B until it coincides with the straight line BC; whereas $\angle$CBA$=-\angle$ABC denotes the equal but opposite rotation from CB to BA. The student should take note of this usage, which will be uniformly employed throughout this book.

The law of the reflection of light, which has been known for more than 2200 years, is contained in the following statement:

The reflected ray lies in the plane of incidence, and the incident and reflected rays make equal angles with the normal on opposite sides of it; that is, $\beta = -\alpha$.

A very accurate experimental proof of this law may be obtained by employing a meridian circle to observe the light reflected from an artificial mercury-horizon, that is, from the horizontal surface of mercury contained in a basin. In fact, this is the actual method used by astronomers in measuring the altitude of a star. The telescope is pointed at the star and then at the image of the star in the mercury mirror, and it will be found that the axis of the telescope in these two observations will be equally inclined to the vertical on opposite sides of it.

A simple lecture-table apparatus for verifying the law of reflection of light consists of a circular disk (Fig. 17) made of ground glass, about one foot or more in diameter, and graduated around the circumference in degrees. This disk is mounted so as to be capable of rotation in a vertical plane about a horizontal axis perpendicular to this plane and passing through the center of the disk. A small piece of a plane mirror B with its plane perpendicular to that of the disk is fastened to the disk at its center, and the mirror is adjusted so that it is perpendicular to the radius BN drawn on the disk. A beam of sunlight falling on the mirror in the direction NB will be reflected back from the mirror in the opposite direction BN, so that in this adjustment of the disk the paths of the incident and reflected rays coincide ($\beta = -\alpha = 0$). Now if the disk is turned so that the incident ray AB makes with the normal BN an angle NBA, the reflected ray will proceed in a direction BC such that $\angle NBC = \angle ABN = -\alpha$.

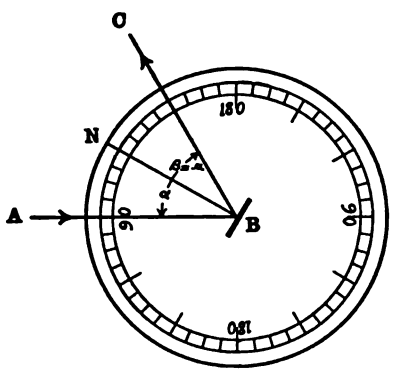


FIG. 17.—Optical disk used to verify law of reflection.

If, without changing the direction of the incident ray, the disk is turned through an angle θ, the plane of the mirror together with the incidence-normal will likewise be turned through this same angle, and the angles of incidence and reflection will each be changed in opposite senses by the amount θ, so that the angle between the incident and reflected rays will be changed by 2θ. Accordingly, *when a plane mirror is turned through a certain angle, the reflected ray will be turned through an angle twice as great.*

14. Huygens's Construction of the Wave-Front in Case of Reflection at a Plane Mirror.

1. *The case of a plane wave reflected from a plane mirror.* The rebound of waves from a polished surface affords a very simple and instructive illustration of HUYGENS's Principle (§ 5). In Fig. 18 the straight line AD represents the trace in the plane of the diagram of a plane mirror, and the straight line AB represents the trace of a portion of the front of an incident plane wave (§ 6) advancing in the direction of the wave-normal BD. At the first instant under consideration the wave-front is supposed to be in the position AB when the disturbance has just reached the point A of the reflecting surface, and from this time forward, according to HUYGENS's theory, the point A is to be regarded as itself a center of disturbance from which secondary hemispherical waves are reflected back into the medium in front of the mirror. Exactly the same state of things will prevail at this instant ($t=0$) at all points of the plane reflecting surface lying on a portion of the straight line perpendicular

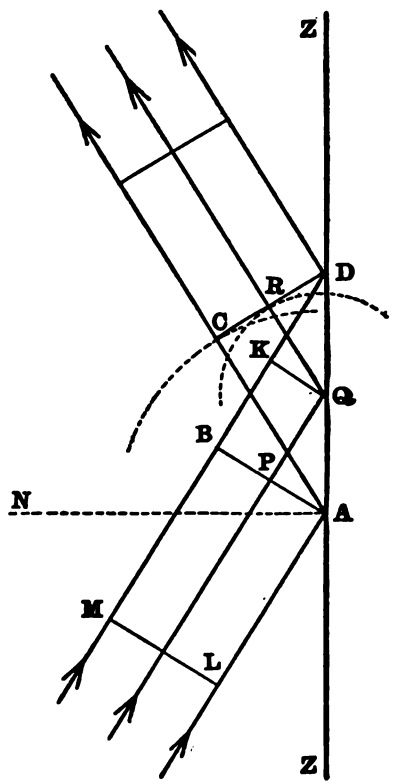


FIG. 18.—HUYGENS's construction of plane wave reflected at plane mirror.

to the plane of the paper at the point A, and the envelop of the hemispherical wavelets originating from these points will be a semicylindrical surface whose axis is the straight line just mentioned. If the speed with which the waves travel is denoted by v, then at the end of the time $t=PQ/v$ the disturbance that was initially at the point P in the wave-front AB will have advanced to a point Q on the reflecting plane between A and D; and from this moment a new set of hemispherical wavelets having their centers all on a straight line perpendicular to the plane of the diagram at the point Q will begin to develop, and their envelop will also be a semicylinder. And so at successively later and later instants the disturbance will arrive in turn at each point along AD; until, finally, after the time $t=BD/v$ the farthermost point D will be reached. Meanwhile, around all the straight lines perpendicular to the plane of the paper at points lying along AD semicylindrical elementary wave-surfaces will have been spreading out from the reflecting surface, the radii of these cylinders diminishing from A towards D. At the time when the disturbance reaches D, the semicylindrical wavelet whose axis passes through A will have expanded until its radius is equal to BD, and at this same instant the semicylindrical wavelet corresponding to a point Q between A and D will have been expanding for a time $(BD—PQ)/v$, and hence its radius will be equal to $(BD—PQ)=(BD—BK)=KD$.

Now, according to HUYGENS's Principle, the surface which at any instant is tangent to all these elementary semicylindrical waves will be the required reflected wave-front at that instant. We shall show that the reflected wave-front is a plane surface which at the moment when the disturbance reaches the point D contains this point; or, what amounts to the same thing, we shall show that if a straight line DC in the plane of the diagram is tangent at C to the semicircle in which this plane cuts the semicylinder whose axis passes through A, it will be a common tangent to all

such semicircles; for example, it will also be tangent to the semicircle in which the plane of the diagram cuts the semicylinder belonging to the point Q. From D draw DC tangent at C to the semicircle described around A as center with radius AC=BD and DR tangent at R to the semicircle described around Q as center with radius QR=KD. The right triangles ABD and ACD are congruent, and hence $\angle DAB = \angle CDA$; and, similarly, in the congruent right triangles QKD and QRD $\angle DQK = \angle RDQ$. But $\angle DQK = \angle DAB$, and therefore $\angle RDQ = \angle CDA$, and hence the two tangents DR and DC coincide. Accordingly, the trace of the reflected wave-front in the plane of the diagram is the straight line CD. This reflected plane wave will be propagated onwards, parallel with itself, in the direction shown by the reflected rays AC, QR, etc. It is evident from the construction that the ray incident at A, the normal AN to the reflecting surface at the incidence-point A, and the reflected ray AC lie all in the same plane; and the equality of the angles of incidence and reflection is an immediate consequence of the congruence of the triangles ABD and ACD.

2. *The case of a spherical wave reflected at a plane mirror.* In Fig. 19 the light is represented as originating from a point-source L and spreading out from it in the form of spherical waves which presently impinge on the plane reflecting surface represented in the diagram by the straight line AD. The nearest point of the reflecting plane to the source at L is the foot A of the perpendicular let fall from L on the straight line AD, and this, therefore, is the first point of the mirror to be affected. Obviously, on account of symmetry with respect to LA, it will be quite sufficient to investigate the procedure of the waves in the plane of the figure. The wave-front at the time the disturbance reaches A will be represented by the arc of a circle described around L as center with radius equal to LA; let P designate the position of a point on this arc, and draw the straight line LP meeting AD at Q. After a time $t = PQ/v$ the dis-

turbance will have advanced from P to Q, and from this moment the point Q will begin to send back wavelets from the reflecting surface. And so in succession one point of the mirror after another will be affected until presently the disturbance reaches the farthest point D. Meanwhile, all the points along AD on one side of AL and along AF on the other side (AF=DA) will have been sending out wavelets whose radii will be greater and greater the nearer these new centers are to the point A midway between D and F. Draw the straight line LD meeting the arc AP in the point B: then at the moment $t=BD/v$ when the disturbance from L has just arrived at D, the reflected wavelet proceeding from A as center will have expanded until its radius is equal to BD, and at this same instant there will also be a wavelet around Q as center of radius (BD—PQ)=(BD—BK)=KD. According to HUYGENS, the problem consists, therefore, in finding the surface which is tangent at a given instant to all these secondary waves. Produce the straight line LA on the other side of the reflecting surface to a point L′ such that AL′=LA, and draw the straight line L′Q, and mark the point R where this straight line produced meets the semicircle described around Q as center with radius KD=QR. Since LQ+QR=LK+KD=LD, obviously,

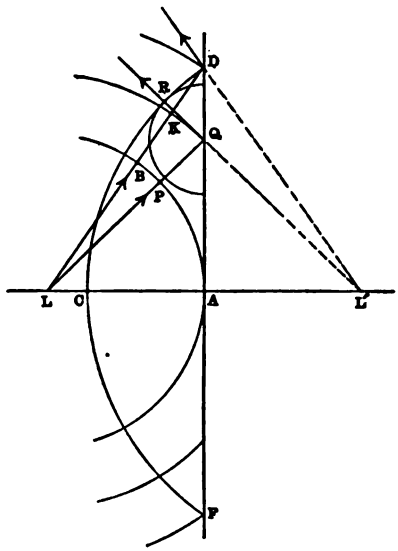


FIG. 19.—HUYGENS's construction of spherical wave reflected at plane mirror.

L'R = L'D; and therefore a circle described around L' as center with radius equal to L'D will touch at R the semicircle described around Q as center with radius equal to QR. Moreover, it will also touch at a point C on the straight line LA the semicircle described around A as center with radius AC = BD. Consequently, this circle will be the envelop of all these semicircles. The reflected wave-front, therefore, is obtained by revolving the arc DCF around LL' as axis. The straight line QR is the path of the reflected ray corresponding to the incident ray PQ; the angle of incidence at Q is equal to the angle ALQ and the angle of reflection is equal to AL'Q, and these angles are evidently equal, in agreement, therefore, with the law of reflection.

15. Image in a Plane Mirror.—In Fig. 19 the plane mirror bisects at right angles the straight line LL', and since the

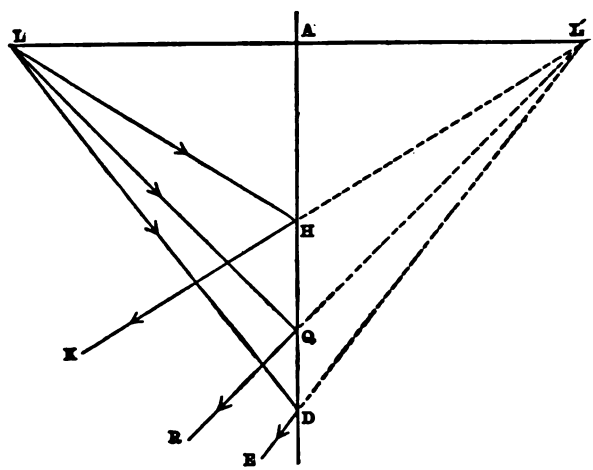


FIG. 20.—L' is image of object-point L in plane mirror AD; AL = L'A.

position of the point L' is independent of the position of the incidence-point Q (Fig. 20), all the rays coming from the luminous point L and falling on the plane mirror will be reflected along paths which, when prolonged backwards,

all meet in the point L′. Thus, *to a homocentric bundle of incident rays reflected at a plane mirror there corresponds also a homocentric bundle of reflected rays.* This remarkable property of converting a homocentric bundle of rays into another homocentric bundle is characteristic of a plane mirror, because no other optical device is capable of it except under conditions that are more or less unrealizable in practice. Thus, the image L′ of an object at L is found by drawing a straight line from L perpendicular to the plane mirror, and producing this line on the other side of the mirror to a point L′ such that the line-segment LL′ is bisected by the plane of the mirror; so that an object in front of a plane mirror is seen in the mirror at the same distance behind it. The image in this case is *virtual* (§ 8). The late Professor SILVANUS THOMPSON in his popular lectures published under the title *Light Visible and Invisible* describes the following simple method of showing how the rays from a candle flame are reflected at a plane mirror (Fig. 21). If a vertical pin mounted on a horizontal base-board is illuminated by a lighted candle, the position of the shadow is determined by the line joining the top of the pin with the source of light. If the pin and the candle are both in front of a plane mirror placed at right angles to the base-board, a second shadow will be cast by the pin on account of the reflected rays from the candle that are intercepted by it, and this shadow will be precisely such as would be produced by a candle flame placed behind the mirror at the place where the image of the actual flame is formed, as may be proved by removing the mirror and transferring the candle to the place where its image was.

If the bundle of incident rays instead of diverging from a point L in front of the plane mirror converged towards a point L behind it (as could easily be effected with the aid of a convergent lens), a *real image* (§ 8) will be produced at a point L′ at the same distance in front of the mirror as the *virtual object-point* L was beyond it.

The image of an extended object is the figure formed by the images of all of its points separately. The diagram (Fig. 22) shows, for example, how an eye at E would see the image L'M' of an object LM reflected in a plane mirror. The series of parallel lines joining corresponding points of

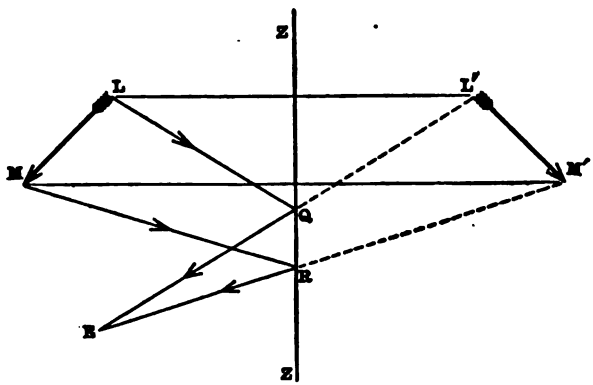


Fig. 22.—Image L'M' of object LM in plane mirror ZZ.

object and image will be bisected at right angles by the plane of the mirror.

The dimensions of the image in a plane mirror are exactly the same as those of the object. Moreover, the top and bottom of the image correspond with the top and bottom of the object, that is, the image is *erect*. Also, the right side of the image corresponds with the right side of the object, and the left side of the image with the left side of the object (Fig. 23), although it is frequently stated in books on optics that when a man stands in front of a mirror the right side of the image shows the left side of the person, and that if the man extends his right hand, the image will extend its left hand. The true explanation of the so-called "perversion" of the image in a plane mirror, which is strikingly seen when a printed page is held in front of the mirror, is that it is *the rear side of the image that is opposite the front*

side of the object. The image of a printed page in a mirror has exactly the same appearance as it would have if the page were held in front of a bright light and it was viewed from behind through the paper. When a person looks in a mirror at his own image, his image appears to be looking back at him in the opposite direction, if he faces east, his image faces west, and if we call the east side of object or image its front side and the west side its rear side, then the rear side of the image is turned towards the front side of the object; although, because this side of the image corresponds to the front side of the object, it is a natural mistake to regard it as also the front side of the image. The explanation of the common impression that, whereas up and down remain unchanged in the image of an object in a plane mirror, right and left are reversed, is probably because a person regarding his own image under such circumstances is unconsciously disposed to transfer himself mentally into coincidence with his image by a rotation of 180°, not around a horizontal, but around a vertical axis, thus producing a confusion of mind as to right and left but not as to top and bottom. The reason why this mental revolution is performed around the vertical axis seems to be due partly to the circumstance that this movement can be readily executed in reality, and partly also perhaps to the fact that the human body happens to be very nearly symmetrical with respect to a vertical plane.

16. The Field of View of a Plane Mirror.—In the adjoining diagram (Fig. 24) the straight line GH represents the trace in the plane of the paper of the surface of a plane mirror, and the point marked O′ shows the position of the center of the pupil of the eye of a person who is supposed to be looking towards the mirror. Evidently, the straight lines HO′, GO′ drawn to O′ from the points G, H in the edge of the mirror will represent the paths of the outermost reflected rays that can enter the eye at O′, and therefore the field of view (§ 9) is limited by the contour of the mirror just

as if the observer were looking into the image-space through a hole in the wall that exactly coincided with the place occupied by the mirror. Corresponding to the pair of reflected rays HO′ and GO′ intersecting at O′ there would be a pair of incident rays directed along the straight lines HO

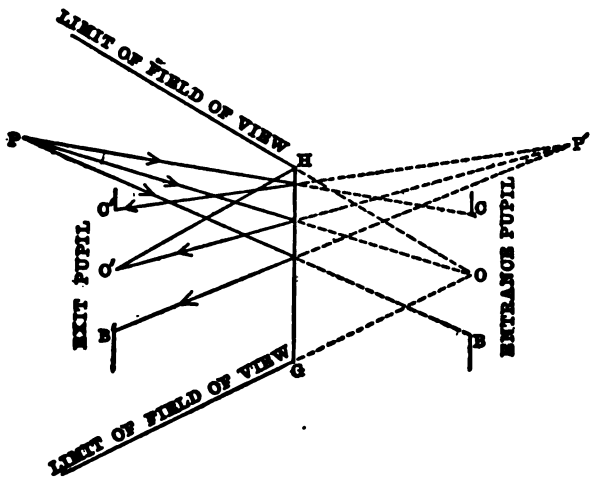


FIG. 24.—Field of view of plane mirror for given position of eye.

and GO towards a point O on the other side of the mirror, and it is evident that O′ will be the real image of a virtual object-point at O (§ 15). Any luminous point lying in front of the plane mirror within the conical surface formed by drawing straight lines such as OG, OH from O to all the points in the edge of the mirror will be visible by reflected light to an eye placed at O′, and hence this cone limits the field of view of the object-space.

Through O′ draw a straight line parallel to GH, and take on it two points C′, B′ at equal distances from O′ on opposite sides of it, and let us suppose that B′C′ represents the diameter in the plane of the diagram of the pupil of the eye. Construct the image BOC of the eye-pupil B′O′C′. Then if P designates the position of a luminous point lying any-

where within the field of view of the object-space, it is clear that the incident rays PO, PC and PB will be reflected at

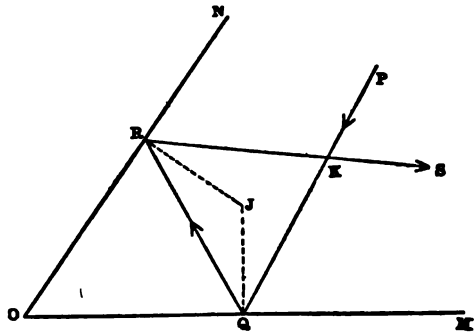


FIG. 25.—Deviation of a ray reflected twice in succession from a pair of inclined mirrors.

the mirror into the pupil of the eye in the directions P'O', P'C' and P'B', as though they had all come from the point P' which is the image of P. This imaginary opening or vir-

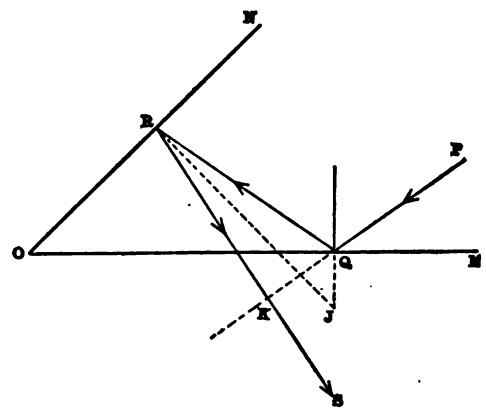


FIG. 26.—Deviation of a ray reflected twice in succession from a pair of inclined mirrors.

tual stop BOC towards which the incident rays must all be directed in order to be reflected into the eye-pupil B'O'C' is

called the *entrance-pupil* of the optical system consisting of the plane mirror and the eye of the observer; and the pupil of the eye itself is called here the *exit-pupil* (see Chapter XII). Since the entrance-pupil limits the apertures of the bundles of rays that ultimately enter the eye, it acts as the *aperture-stop* of the system (§ 11).

17. Successive Reflections from two Plane Mirrors.—Any section made by a plane perpendicular to the line of intersection of the planes of a pair of inclined mirrors is called a *principal section* of the system. *If a ray lying in a principal section is reflected successively at two plane mirrors, it will be deviated from its original direction by an angle equal to twice the dihedral angle between the mirrors.*

Let the plane of the principal section intersect the planes of the mirrors in the straight lines OM, ON (Figs. 25 and 26); and let $\gamma = \angle MON$ denote the angle between the mirrors. The ray PQ lying in the plane MON is incident on the mirror OM at the point Q, whence it is reflected along the straight line QR, meeting the mirror ON at the point R, where it is again reflected, proceeding in the direction RS. Let the point of intersection of the straight lines PQ and RS be designated by K. Then $\angle PKR$ is the angle between the original direction of the ray and its direction after undergoing two reflections, and we must show that this angle is equal to 2γ.

Draw the incidence-normals at Q and R, and prolong them until they meet at J. Then by the law of reflection the straight lines QJ and RJ bisect the angles PQR and QRS, respectively.

In Fig. 25, $\angle PKR = \angle PQR + \angle QRS = 2(\angle JQR + \angle QRJ)$
$= 2(180° - \angle RJQ) = 2\gamma;$

and in Fig. 26, $\angle PKR = \angle PQR - \angle SRQ$
$= 2(180° - \angle RQJ - \angle JRQ)$
$= 2\angle QJR = 2\gamma.$

18. Images in a System of Two Inclined Mirrors.—When a luminous point lies in the dihedral angle between two

plane mirrors, some of its rays will fall on one mirror and some on the other, and consequently there will be two sets of images. In Fig. 27 the plane of the diagram is the principal section which contains the point-source S, and the straight lines OM, ON represent the traces of the mirrors

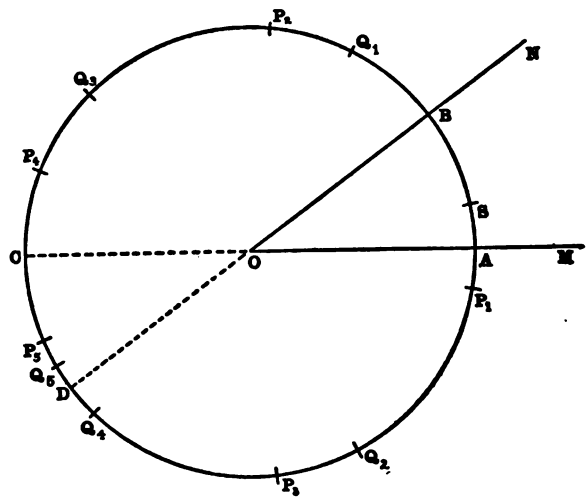


FIG. 27.—Images of a luminous point S in a pair of inclined mirrors OM and ON.

in this plane. The rays which fall first on the mirror OM will be reflected as though they came from the image P_1 of the luminous point S in this mirror. Some of these rays falling on the mirror ON will be again reflected and proceed thence as though they came from the point P_2 which is the image of P_1 in the mirror ON. Thus, by successive reflections, first at one of the mirrors and then at the other, a series of images P_1, P_2, etc., will be formed by those rays which fall first on the mirror OM; let us call this the P-series of images. Similarly, the rays that fall first on the mirror ON will produce another series of images Q_1, Q_2, etc., which will be called the Q-series. Each of these series will terminate with an image which lies behind both mirrors in the

dihedral angle COD opposite the angle MON between the mirrors themselves; because rays which, after reflection at one of the mirrors, appear to come from a point thus situated cannot fall on the other mirror, and so there will be no more images after this one.

Since the straight line OM is the perpendicular bisector of the line-segment SP_1, the points S and P_1 are equidistant from every point in the straight line OM; and, similarly, since P_2 is the image of P_1 in the plane mirror ON, these two points are likewise equidistant from every point in the straight line ON. Accordingly, the three points S, P_1, P_2 are all equidistant from the point O where the straight lines OM and ON intersect. Applying the same reasoning to all the other images, we perceive that *the images of both series are ranged on the circumference of a circle whose center is at* O *and whose radius is* OS.

In the following discussion of the angular distances of the images from the luminous point S, the angles will be reckoned always in the same sense, either all clockwise or all counter-clockwise. Let $\gamma = \angle AOB$ denote the angle between the two mirrors, the letters A and B referring to the points where the circle crosses the planes of the mirrors OM and ON, respectively. Also, let $\alpha = \angle AOS$, $\beta = \angle SOB$ denote the angular distances of S from A, B, respectively, so that $\alpha + \beta = \gamma$. Then

$$\angle P_1OS = 2\alpha;$$

$$\angle SOP_2 = \angle SOB + \angle BOP_2 = \beta + \angle P_1OB = 2(\alpha+\beta) = 2\gamma;$$

$$\angle P_3OS = \angle P_3OA + \alpha = \angle AOP_2 + \alpha = \angle SOP_2 + 2\alpha = 2\gamma + 2\alpha;$$

$$\angle SOP_4 = \angle SOB + \angle BOP_4 = \beta + \angle P_3OB = 2\beta + \angle P_3OS = 2(\alpha+\beta+\gamma) = 4\gamma;$$

$$\angle P_5OS = \angle P_5OA + \alpha = \angle AOP_4 + \alpha = \angle SOP_4 + 2\alpha = 4\gamma + 2\alpha.$$

In general, therefore,

$$\angle SOP_{2k} = 2k\gamma, \qquad \angle P_{2k+1}OS = 2k\gamma + 2\alpha,$$

where P_{2k}, P_{2k+1} designate the positions of the $2k$th and

$(2k+1)$th images of the P-series, k denoting any integer, and where the angles SOP_{2k}, $P_{2k+1}OS$ are the angles subtended by the arcs SBP_{2k}, $P_{2k+1}AS$, respectively. Similarly, for the Q-series of images we find:

$$\angle Q_{2k}OS = 2k\gamma, \qquad \angle SOQ_{2k+1} = 2k\gamma + 2\beta,$$

where these angles are the angles subtended by the arcs $Q_{2k}AS$, SBQ_{2k+1}, respectively.

Evidently, the image P_{2k+1} will fall on the arc CD behind both mirrors, if arc $P_{2k+1}AS$ > arc DAS, that is, if

$$2k\gamma + 2\alpha > 180° - \beta;$$

and, by adding $(\beta - \alpha)$ to both sides of this inequation, and dividing through by γ, this condition may be expressed as follows:

$$2k+1 > \frac{180° - \alpha}{\gamma}.$$

In the same way we find that the image P_{2k} will fall between C and D if

$$2k > \frac{180° - \alpha}{\gamma}.$$

Thus, *the total number of images of the* P-*series, whether it be odd or even, will be given by the integer next higher than* $(180° - \alpha)/\gamma$; and, similarly, *the total number of images of the* Q-*series will be given by the integer next higher than* $(180° - \beta)/\gamma$.

The only exception to this rule is when the angle γ is contained in $(180° - \alpha)$ or $(180° - \beta)$ an exact whole number of times; in the former case the last image of the P-series falls at C, and in the latter case the last image of the Q-series falls at D; and instead of taking the integers next above the quotient $(180° - \alpha)/\gamma$ or $(180° - \beta)/\gamma$, we must take the actual integer obtained by the division. An example will make the matter clear. Thus, suppose $\gamma = 27°$, $\alpha = 8°$, then $\beta = 19°$, and the integers next higher than $(180° - \alpha)/\gamma$ and $(180° - \beta)/\gamma$ will be 7 and 6, respectively; hence in this case there will be 7 images of the P-series and 6 images o the Q-series or 13 images in all. But if $\alpha = 10°$ and $\beta = 17°$,

each series will be found to have 7 images, 14 images in all. The exceptional case occurs when $\alpha = 9°$ and $\beta = 18°$, for then $(180° - \beta)/\gamma = 6$, and hence there will be 7 P-images and 6 Q-images.

If the angle γ between the mirrors is an exact multiple of 180°, that is, if $180°/\gamma = p$, where p denotes an integer, the integers next higher than $(180° - \alpha)/\gamma$ and $(180° - \beta)/\gamma$ will both be equal to p, no matter what may be the special position of the object between the two mirrors; so that in such a case the number of images in each series will be equal, but the last image of one set will coincide with the last of the other. In fact, the points S, P_2, P_4, . . . Q_4, Q_2 and the points P_1, P_3, . . . Q_3, Q_1 are the vertices of two equal regular polygons, of p sides each; and if p is odd, the polygon $P_1P_3 \ldots Q_3Q_1$ will have one of its corners between C and D, whereas if p is even, one of the corners of the polygon $SP_2P_4 \ldots Q_4Q_2$ will fall between C and D; in either case this vertex is the position of the last image of both series. Thus, for example, if $\gamma = 60°$ (Fig. 28), then $p = 3$, and the two polygons are the equilateral triangles SP_2Q_2 and $P_1P_3Q_1$ (or $P_1Q_3Q_1$).

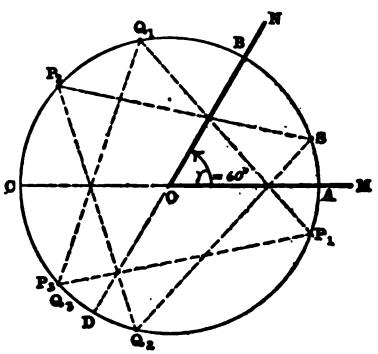


FIG. 28.—Images of a luminous point in a pair of plane mirrors inclined to each other at an angle of 60°.

The toy called a *kaleidoscope*, devised by Sir DAVID BREWSTER (1781–1868), consists essentially of two long narrow strips of mirror-glass inclined to each other at an angle of 60° and inclosed in a cylindrical tube. One end of the tube is closed by a circular piece of ground glass whereon are loosely disposed a lot of fragments of colored glass or

beads, and at the other end of the tube there is a peep-hole. When the instrument is held towards the light, an observer looking in it will see an exquisitely beautiful and symmetrical pattern formed by the colored objects and their images, the form of which may be almost endlessly varied by revolving the tube around its axis so that the bits of glass assume new

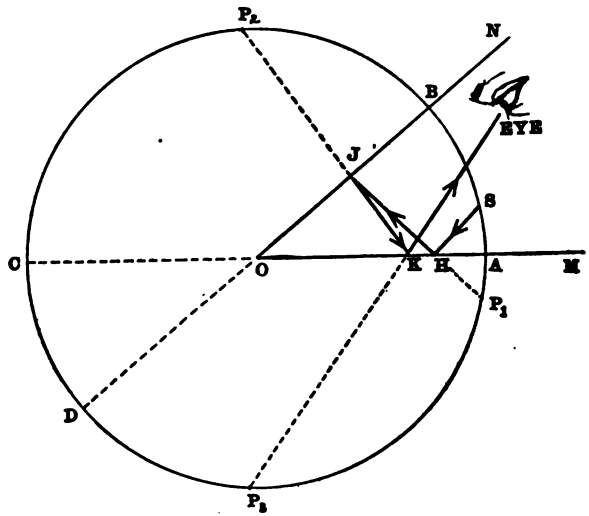


FIG. 29.—Path of ray reflected into eye from a pair of inclined mirrors.

positions. In fact, this device has been turned to practical use in making designs for carpets and wall-papers.

19. Construction of the Path of a Ray Reflected into the Eye from a Pair of Inclined Mirrors.—In order to trace the paths of the rays by which a spectator standing in front of a pair of inclined mirrors sees the image of a luminous point, it is convenient to assume, for the sake of simplicity, that the eye at E in Fig. 29 lies in the plane of the paper. The first step in the construction of the path of the ray is to draw the straight line from the given image-point to the eye, because if the eye sees this point the light that enters the

eye must arrive along this line. If this line does not cross the mirror in which the image is produced, this particular image will not be visible from the point E. Now join the point where this line meets the mirror with the preceding image in the same series; the part of this line that lies between the two mirrors will evidently show the route of the

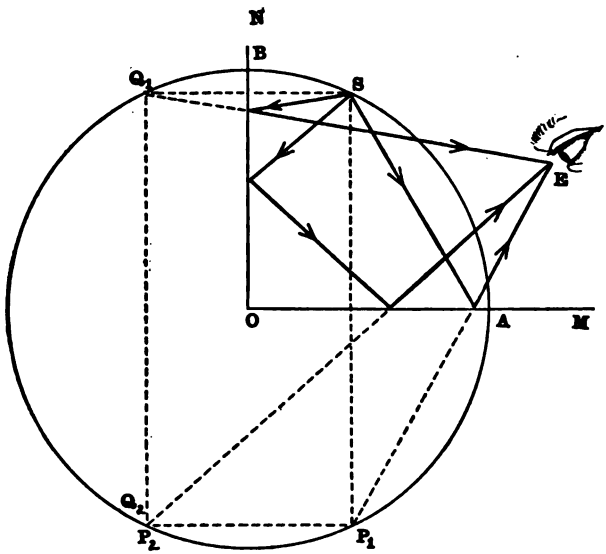


FIG. 30.—Showing how an eye at E sees the images of a luminous point S in a rectangular pair of plane mirrors.

light before its last reflection. Proceeding in this fashion from one mirror to the other, we shall trace backwards the zigzag path of the ray until we arrive finally at the luminous source at S. Consider, for example, the image P_3 formed in the mirror OA in Fig. 29. This image is visible to the eye at E because the straight line P_3E cuts at K the mirror OA. If J and H designate the points where the straight lines KP_2 and JP_1 meet the mirrors OB and OA, respectively, the broken line SHJKE will represent the path of the ray from the source at S into the eye at E. Fig. 30 shows how

an eye at E in front of two perpendicular plane mirrors can see the images P_1, P_2, Q_1 and Q_2.

20. Rectangular Combinations of Plane Mirrors.—In a rectangular combination of two plane mirrors ($\gamma = 90°$) the image formed by two successive reflections will be inverted in the principal section of the system, but in any plane at right angles to a principal section the image will be erect. For example, if an object is placed in front of two vertical plane mirrors at right angles to each other, the image produced by two reflections will have the same position and appearance as if the object had been revolved bodily through an angle of 180° about a vertical axis coinciding with the line of intersection of the planes of the mirrors, as represented in Fig. 31. In this case the image remains vertically erect, whereas it is horizontally inverted. On the other hand, if one of the mirrors is vertical and the other horizontal, the image by twofold reflection will have the same position and appearance as if the object had been revolved through 180° around a horizontal axis coinciding with the line of intersection of the two mirrors (Fig. 32); that is, the image now will be upside down but not inverted horizontally.

Therefore, *in order to obtain an image that is completely reversed in every respect*, two rectangular combinations of plane mirrors may be employed with their principal sections mutually at right angles, so disposed that the rays coming from the object will be reflected in succession from each of the four plane surfaces. An auxiliary system of this description is sometimes used in connection with an optical instrument for the purpose of *rectifying the image* which otherwise would be seen inverted. A rectifying device depending on this principle is the so-called PORRO prism-system (1852), utilized by ABBE (1840–1905) in the design of the famous prism binocular telescope or field-glasses (*c.* 1883). A sketch of the arrangement is shown in Fig. 33. Two rectangular prisms are placed in the tube of the instrument, between the objective and the

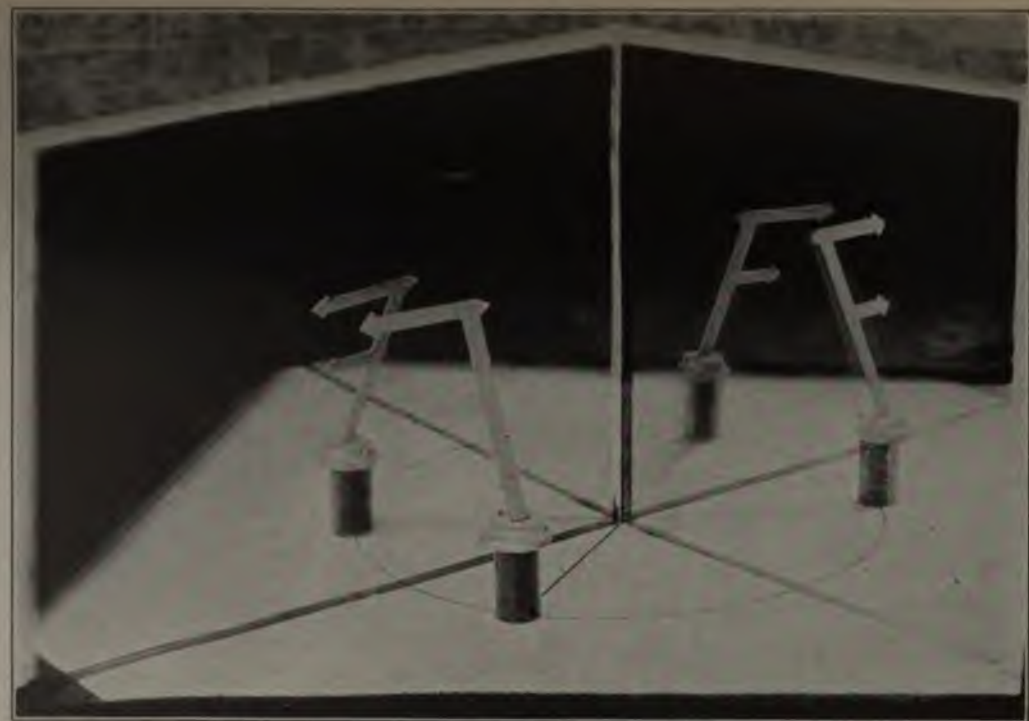


FIG. 31.—Image of an object in a rectangular pair of plane mirrors (from actual photograph); showing how the last image is obtained by rotating the object through 180° around the line of intersection of the mirrors. Both mirrors in vertical planes.

21. Applications of the Plane Mirror.—It is hardly necessary to say that the plane mirror for various purposes has been in use among civilized peoples of all ages; although the use of mirrors as articles of household furniture and decoration does not go back farther than the early part of the 16th century. By a combination of two or more plane mirrors a lady can arrange the back of

FIG. 34.—Porte lumière.

her dress and in fact see herself as others see her. With the aid of a mirror or combination of mirrors many ingenious "magical effects" are produced in theaters. The plane mirror also constitutes an essential part of numerous useful scientific instruments in some of which its only duty is to alter the course of a beam of light, whereas in various forms of goniometrical instruments and contrivances for determining an angular magnitude that is not easily measured

directly the angle in question is ascertained indirectly by observing the angle turned through by a ray of light which is reflected from a plane mirror.

22. Porte Lumière and Heliostat.—As good an illustration as can be given of the use of a plane mirror for changing the direction of a beam of sunlight is afforded by the

Fig. 35.—Heliostat.

porte lumière (Fig. 34), which consists essentially of a plane mirror ingeniously mounted so as to be capable of rotation about two rectangular axes, whereby it may be readily adjusted in any desired azimuth and reflect a beam of sunlight through a suitable opening in the wall of the building to any part of the interior of the room.

However, owing to the diurnal movement of the sun,

a continual adjustment of the mirror is necessary in order to keep the spot of light for any length of time at the place in the room where it is needed, and sometimes this manipulation is very inconvenient and annoying, especially in the case of a laboratory experiment extending perhaps over a considerable part of a day. Thus, for example, in studying the solar spectrum it is often desirable to illuminate the slit in the collimator tube of the spectrometer for hours at a time. For such purposes it is better to use a *heliostat* (Fig. 35), which is contrived so that the plane mirror is continuously revolved by clockwork around an axis parallel to the earth's axis so as to preserve always the same relative position with respect to the sun in its apparent diurnal motion in the sky. The mirror can also be turned about a horizontal axis, and it has first to be adjusted about this axis so that the rays of the sun are reflected towards the north pole of the celestial sphere, that is, parallel to the axis of the earth. The mirror being adjusted at this angle, which will depend on the declination of the sun above or below the celestial equator, and turning at the rate of 15° per hour around an axis parallel to the axis of rotation of the earth, it is evident that the rays of the sun will continue to be reflected constantly in the same direction. Suppose, for example, that the mirror is adjusted in the position ZZ (Fig. 36) so that the ray SB coming from the sun at S is reflected at B in the direction BP parallel to the axis of the earth and therefore parallel to the axis of rotation AB of the mirror. If the polar distance of the sun is denoted by $2\alpha = \angle PBS$, and if the angle between the normal to the mirror and the axis of rotation is denoted by η, then, evidently, $\eta = \alpha$. If the sun's declination on a certain day

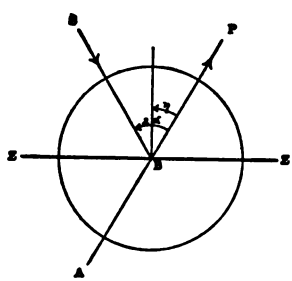


FIG. 36.—Principle of heliostat.

is $+10°$, then $2\alpha = 90° - 10° = 80°$, and $\eta = 40°$. If, on the other hand, the sun is 10° below the equator, $2\alpha = 100°$ and $\eta = 50°$.

The heliostat is provided also with a fixed mirror which reflects the rays from the rotating mirror in a definite direction, as desired, usually in a horizontal direction into the room where the sunlight is to be used. Generally, the instrument is mounted on a permanent ledge outside the window; sometimes it is placed on the roof of the building and the fixed mirror adjusted so as to send the sun's rays down a vertical tube at the bottom of which there is another mirror placed at an angle of 45° with the vertical where the rays are once more reflected so that the beam of sunlight which enters the room will be horizontal.

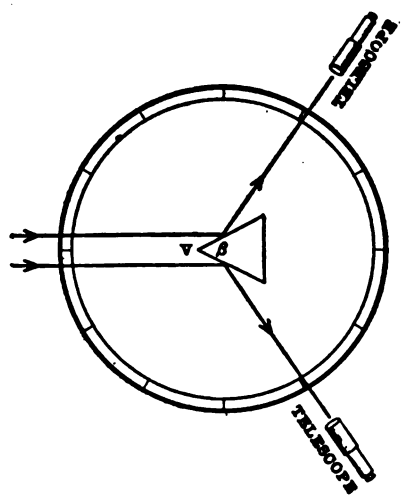


FIG. 37.—Measurement of angle of prism.

23. Measurement of the Angle of a Prism.—Another laboratory application of the principle of a plane mirror is seen in the method of using a goniometer to ascertain the dihedral angle between two plane faces of a glass prism (§ 48). The angle that is actually measured by the goniometer is the angular distance between the images of a distant object as seen in the two faces of the prism. Parallel rays coming from a far-off source at S (Fig. 37) and incident on the two faces of the prism that meet in the edge V are reflected as shown in the diagram, and the angle between the two directions of the reflected rays is obviously equal to twice the dihedral angle β.

24. Measure of Angular Deflections by Mirror and Scale.—The angular rotation of a body, for example, the deflection of the magnetic needle of a galvanometer, is frequently measured by attaching a mirror to the rotating body from which a beam of light is reflected. This reflected light acts as a long weightless pointer whereby the actual

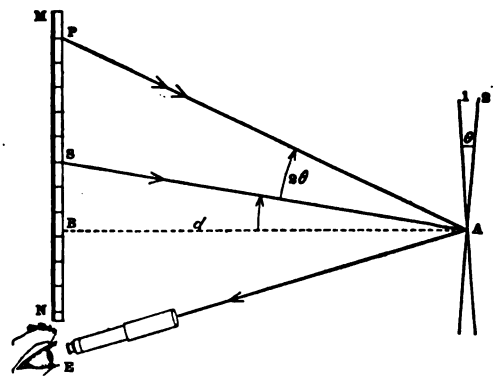


FIG. 38.—Mirror, telescope and scale for measurement of angles.

movement of the body can be magnified to any extent without in the least affecting the sensitiveness of the apparatus.

In Fig. 38 the plane mirror which is capable of rotation about an axis perpendicular at A to the plane of the paper is represented in its initial position by the line-segment marked 1. The straight line MN in front of the mirror and at a known distance (d=AB) from it represents a scale graduated in equal divisions. An eye at E looking through a telescope pointed towards A will see the image in the mirror of the scale-division at S, the so-called "zero-reading," because the light from S incident at A on the mirror in the position 1 ("equilibrium-position") is reflected along AE into the eye at E. If now the mirror is turned through an angle θ into the position marked 2, another scale-division will come into the field of view of the telescope and coin-

cide with the cross-hair in the eye-piece. If this scale-division corresponds to the point marked P, it is the light that comes along PA that is now reflected along AE into the eye at E; and evidently, according to § 13, $\angle \mathrm{PAS} = 2\theta$. In making a measurement by this method, the three points designated by S, B and E are generally adjusted so as to be very near together, if not actually coincident. If they were coincident, the planes of the mirror and scale would be parallel, and the axis of the telescope would coincide with the straight line BA perpendicular to the scale at B. But in any case the $\angle \mathrm{BAS} = \epsilon$ will be a constant, depending partly on the initial position of the mirror and partly on the direction of the axis of the telescope; thus,

$$\tan \epsilon = a/d,$$

where $a = \mathrm{BS}$. If, therefore, we put $x = \mathrm{SP}$, we have:

$$\frac{x}{d} = \tan(\epsilon + 2\theta) - \tan \epsilon;$$

whence, since the value of x can be read off on the scale, it will be easy to calculate the value of the required angle θ through which the mirror has been turned. In many cases where this method is employed the angles denoted by θ and ϵ are both so small that there will be little error in substituting the angles themselves in place of their tangents. Under these circumstances the above formula will be greatly simplified, for the angle ϵ will disappear entirely, and we shall obtain:

$$\theta = \frac{x}{2d},$$

where, however, it must be noted that this expression gives the value of the angle θ in radians. The value of θ in degrees is found by multiplying the right-hand side of this formula by $180/\pi$, so that we obtain:

$$\theta = \frac{90.x}{\pi.d} \text{ degrees.}$$

A lamp and scale is sometimes used instead of a telescope and scale, the light of the lamp being reflected from the

mirror on to the scale which is usually made of translucent glass, so that it is easy to read the position of the spot of light.

25. Hadley's Sextant.—Another instrument which utilizes the principle of § 17 is the sextant, which is employed for

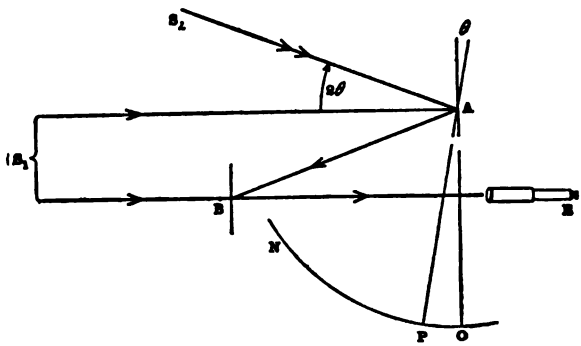


FIG. 39.—Principle of sextant.

measuring the angular distance between two bodies, for example, the altitude of the sun above the sea-horizon. The plan and essential features of this apparatus are shown in Fig. 39. At the center A of a graduated circular arc ON a small mirror is set up in a plane at right angles to that of the arc. This mirror can be turned about an axis perpendicular to the plane of the paper and passing through A. Rigidly connected to this mirror and turning with it is a long solid arm AP whose other end P, provided with a vernier scale, moves over the arc ON, whereby the angle through which the mirror turns can be accurately measured. A little beyond the extremity N of the graduated part of the arc, a second mirror B is erected facing the first mirror. The plane of this mirror is likewise perpendicular to that of the circle, but from the upper half of it the silver has been removed, so that this portion of the mirror B is transparent. Moreover, this mirror is fixed with respect to the instrument. An observer looking through a peep-hole or tele-

scope attached to the instrument towards the mirror B may see a distant object through the upper transparent part of this mirror, and at the same time he may also see just below it the image of a second object reflected in the lower half of the glass. When the planes of the two mirrors A and B are parallel, the zero-mark of the vernier on the movable arm coincides with the zero-mark O of the graduated arc. Suppose, for example, that when the two mirrors are parallel to each other, the instrument is pointed at a distant

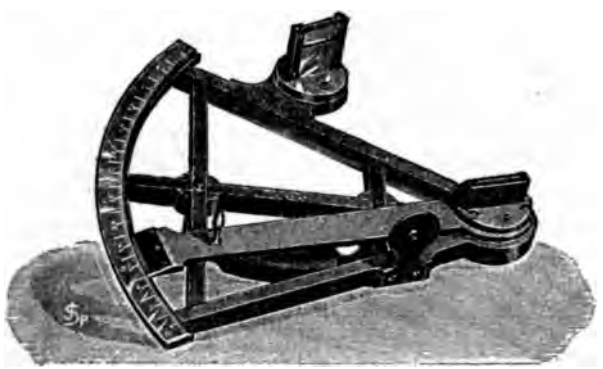


FIG. 40.—Model of mirror sextant.

object, say, a star at S_1, which will be seen directly through the upper half of the fixed mirror B. At the same time the observer will see an image of the object S_1 by rays which have been reflected from the mirror A to the mirror B and thence into the eye at E; for if the two mirrors are parallel, the direction of a ray after two reflections will be the same as its initial direction. If now the mirror A is turned until the image of another object at S_2 comes into the field of vision, the two objects S_1 and S_2 will be seen simultaneously, for with the mirror at A in this new position the incident ray S_2A will be the ray that is reflected from A to B and thence, as before, into the eye at E. Moreover, since the angle between the original direction S_2A of this ray and its final direction

S_1A is equal to double the angle between the planes of the two mirrors, that is, is equal to 2θ, where $\theta = \angle OAP$, the angular distance between the objects at S_1 and S_2 must be equal to 2θ, that is, $\angle S_1AS_2 = 2\theta$. In order to save trouble in making the readings, half-degrees on the graduated arc are reckoned as degrees, so that the value of the angle 2θ is read directly on the scale. As the angular distance between the objects will seldom exceed 120°, and since, in fact, the method is not very accurate for angles greater than this, the actual length of the graduated arc need not be greater than about 60° or one-sixth of the circumference; whence the name sextant is derived.

A simple model of a mirror sextant is shown in Fig. 40. For accurate measurements the instrument is made of metal with a scale etched on a silver strip. Moreover, a telescope is used instead of a peep-hole; so that with a fine sextant it is comparatively easy to measure the angular distance between two points to within one-half minute of arc. One great advantage of this instrument is its portability, and since it does not have to be mounted on a stand, it is very serviceable on shipboard for measurements of altitude and determinations of latitude, etc.

PROBLEMS

1. The top of a vertical plane mirror 2 feet high is 4 feet from the floor. The eye of a person standing in front of the mirror is 6 feet from the floor and 3 feet from the mirror. What are the distances from the wall on which the mirror hangs of the farthest and nearest points on the floor that are visible in the mirror? Ans. 6 ft.; 18 in.

2. A ray of light is reflected at a plane mirror. Show that if the mirror is turned through an angle θ, the reflected ray will be turned through an angle 2θ.

3. Show that the deviation of a ray reflected once at each of two plane mirrors is equal to twice the angle between the mirrors.

4. If a plane mirror is turned through an angle of 5°, what is the deflection indicated by the reading on a straight scale 100 cm. from the mirror? Ans. About 17.6 cm.

5. Find the angle turned through by the mirror when the deflection on the scale in the preceding example is 10 cm.? Ans. 2° 52′.

6. What must be the length of a vertical plane mirror in order that a man standing in front of it may see a full length image of himself? Ans. The length of the mirror must be equal to half the height of the man.

7. Show that a plane mirror bisects at right angles the line joining an object-point with its image.

8. A ray of light proceeding from a point A is reflected from a plane mirror to a point B. Show that the path pursued by the light is shorter than any other path from A to the mirror and thence to B.

9. Give HUYGENS's construction, (1) for the reflection of a plane wave at a plane mirror, and (2) for the reflection of a spherical wave at a plane mirror.

10. Explain clearly how to determine the limits of the field of view in a plane mirror for a given position of the eye of the spectator.

11. A candle is placed between two parallel plane mirrors. Show how an observer can see the image of the candle produced by rays which have been twice reflected at one mirror and three times at the other. Draw accurate diagram showing the paths of the rays, the positions of the images, etc.; and give clear explanation of the figure.

12. OA and OB are two plane mirrors inclined at an angle of 15°, and P is a point in OA. At what angle must a ray of light from P be incident on OB in order that after three reflections it may be parallel to OA? Ans. 45°.

13. Show that the image of a luminous object placed between two plane mirrors all lie on a circle.

14. Show how by means of two plane mirrors a man standing in front of one of them can see the image of the back

of his head. Trace the course of the rays from the back of his head into his eye and explain clearly.

15. Show by a diagram, with clear explanations, how one sees the image of an arrow in a plane mirror.

16. Construct the image of an arrow formed by two reflections in a pair of inclined mirrors, (1) when the mirrors are at right angles, and (2) when the angle between the mirrors is 60°.

17. Show how a horizontal shadow of a vertical rod can be thrown on a vertical screen by a point-source of light with the aid of a plane mirror. Draw a diagram.

18. An object is placed between two plane mirrors inclined at an angle of 45°. Show by a figure how a spectator may see the image after four successive reflections. Give clear explanation.

19. Two plane mirrors are inclined at an angle of 50°. Show that there will be 7 or 8 images of a luminous point placed between them, according as its angular distance from the nearer mirror is or is not less than 20°.

20. Find the number of images formed when a bright point is placed between two plane mirrors inclined to each other at an angle of 25°. Ans. 14 or 15 images according as the angular distance of the luminous point from the nearer mirror is or is not less than 5°.

21. A luminous object moves about between two plane mirrors, which are inclined at an angle of 27°. Prove that at any moment the number of images is 13 or 14 according as the angular distance of the luminous point from the nearer mirror is or is not less than 9°.

22. The angle between a pair of inclined mirrors is 80°. Find the position of an object which is reproduced by 5 images. Ans. The object must be less than 20° from the nearer mirror.

23. Describe a sextant with the aid of a diagram, and explain its use.

24. Describe and explain the heliostat.

25. Construct the image of the capital letter F as seen in a plane mirror.

26. When a candle-flame is placed in front of a screen with a pin-hole opening, an image of the flame is formed on a second screen placed parallel to the first. But if the second screen is replaced by a plane mirror, the image will be formed on the back of the first screen. Explain how this happens.

27. Explain clearly (with diagram, formula, etc.) the method of using a mirror and scale for measurement of angles.

28. Describe how the dihedral angle of a glass prism is measured on a goniometer-circle.

CHAPTER III

REFRACTION OF LIGHT

26. Passage of Light from One Medium to Another.—Hardly any one can have failed to observe that the course of light in passing obliquely from water to air is abruptly changed at the surface of the water. For example, if a coin is placed at A in the bottom of a china bowl (Fig. 41), and if the eye is adjusted at a point C so that the coin is hid from view by the side of the vessel, then, without altering the position of the eye, the coin can be made visible merely by pouring water in the bowl up to a certain level.

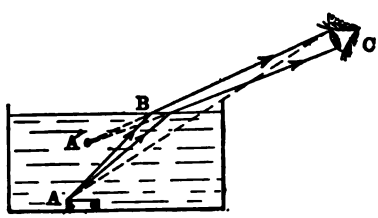


FIG. 41.—Coin at bottom of bowl rendered visible by filling bowl with water.

The broken line ACB illustrates how a ray proceeding from A may be bent at the surface of the water so as to pass over the edge of the bowl and enter the eye at C. It is true the coin will will not appear to be at A but at a point A′ nearer the surface of the water and displaced a little sideways towards the eye, because the rays that come to the eye intersect at this point A′ (§ 42). A clear pool of water seems to be shallower than it really is, and this illusion is greater in proportion as the line of sight is more oblique, so that bright objects at the bottom of the pool appear to be crowded together towards the surface. When a stick is partly immersed in water, the part under water appears to be bent up towards the surface (§ 42).

This bending of the rays which takes place when light crosses the boundary between two media is called *refraction*.

The path of a beam of sunlight through water can easily be shown by mixing a little milk in the water or by stirring in it a minute quantity of chalk-dust, while a puff of smoke will at once reveal the track of the beam in the air, so that the phenomena of refraction can readily be exhibited to the eye. In every case it will be found that the ray is bent farther from the incidence-normal in the rarer or less dense medium (see § 30); and here also, as in the case of reflection, there is a perfectly definite connection between the direction of the incident ray and that of the corresponding refracted ray.

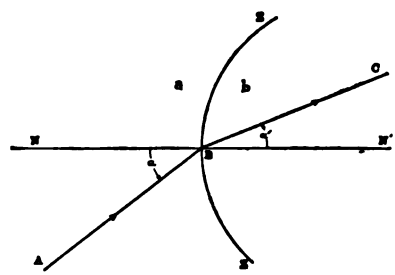


FIG. 42.—Law of Refraction.

27. Law of Refraction.—In Fig. 42 the straight line AB represents the path of a ray incident at the point B on a smooth refracting surface separating two media which for the present will be designated by the letters a and b. The straight line NN′ drawn perpendicular to the plane which is tangent to the refracting surface at B represents the incidence-normal; and the plane of the paper which contains the incident ray and the incidence-normal is the plane of incidence, as already defined (§ 13). The line ZZ represents the trace of the refracting surface in this plane. And, finally, the path of the refracted ray is shown by the straight line BC. The angles of incidence and refraction are defined to be the acute angles through which the incidence-normal has to be turned in order to bring it into coincidence with the incident and refracted rays, respectively. Thus, if these angles are denoted by α, α', then

$$\alpha = \angle \mathrm{NBA}, \qquad \alpha' = \angle \mathrm{N'BC}.$$

In the figure as drawn the angle α is represented as greater than the angle α', so that, according to the statement at

the end of § 26, the medium a is less dense or "rarer" than the medium b.

Before stating the relation which is found to exist between the angles α and α', it is necessary to allude to NEWTON'S great discovery that sunlight and indeed so-called "white light" of any kind, as, for example, the light of an arc lamp, is composed of light of an innumerable variety of colors (see Chapter XIV), as may be shown by passing a beam of sunlight through a glass prism, whereby it will be seen that white light is a mixture of all the colors of the spectrum in all their infinite varieties of hues. On the other hand, *monochromatic light*, as it is called, is light of some one definite color, as, for example, the yellow light emitted by a sodium flame which may be obtained by burning common salt in the flame of a BUNSEN burner. In geometrical optics, unless we are specially concerned with the investigation of color-phenomena (as in Chapter XIV), it is nearly always tacitly assumed that the source of the light is monochromatic.

The *law of refraction*, as found by experiment, may now be stated as follows:

The refracted ray lies in the plane of incidence on the opposite side of the normal in the second medium from the incident ray in the first medium; and the sines of the angles of incidence and refraction are to each other in a constant ratio, the value of which depends only on the nature of the two media and on the color (or wave-length) of the light.

This constant ratio, denoted by the symbol n_{ab}, is called *the relative index of refraction from the first medium* (a) *to the second medium* (b) *for light of the given color;* thus,

$$\frac{\sin \alpha}{\sin \alpha'} = n_{ab};$$

the value of this constant, as a rule, being greatest for violet and least for red light, so that the violet rays are the most "refrangible" of all. When light is refracted from air (a) to water (w) the relative index of refraction is, approximately,

$n_{aw}=4/3$, and hence under these circumstances $\sin a' = \frac{3}{4} \sin a$. Although there are many different varieties of optical glass, for rough calculations the value of the relative index of refraction from air (a) to glass (g) may be taken as $n_{ag}=3/2$; which means that the sine of the angle which the ray makes with the normal in glass is about two-thirds of the sine of the angle which the corresponding ray makes with the normal in air.

Although the law of refraction is quite simple, it somehow eluded discovery until early in the seventeenth century when the true relation between the angle of refraction and the angle of incidence was first ascertained by WILLEBRORD SNELL (1591–1626) or SNELLIUS, of Leyden, and the law is, therefore, often referred to as SNELL'S Law of Refraction. The law was first published by the French philosopher DESCARTES (1596–1650), who had probably seen SNELL'S papers, although he does not allude to him by name.

28. Experimental Proof of the Law of Refraction.—The relation between the angles of incidence and refraction can be very strikingly exhibited with the aid of the optical disk that was mentioned in § 13 in connection with a lecture-table experiment for verifying the law of reflection of light. The vertical ground glass disk is adjusted in the track of a narrow beam of sunlight (or parallel rays from a lantern) in such a position that the path of the light is shown by a band of light crossing the face of the disk along one of its diameters. The glass body through which the light is refracted has the form of a semicylinder, the two plane parallel sides being ground rough so as to be more or less opaque, whereas the curved surface and the diametral plane face are both highly polished. This half-disk has a radius of about 2 inches and is about one-half inch thick or more. It can be fastened against the vertical face of the optical disk with its axis horizontal and coinciding with the axis of rotation of the disk, as represented in the diagram Fig. 43. If this adjustment is made, and the disk turned so that the inci-

dent ray AB meets the polished plane face of the glass body at its center B, the refracted ray BC will proceed through the glass along a radius of the semicylinder, and therefore meeting the curved surface normally, it will emerge again into the air without being further deviated. The diameter NN′ which is marked on the face of the optical disk is normal to the plane surface of the glass body, and if from the points A and C where the incident and refracted rays cross the circumference of the disk perpendiculars are let fall on the normal NN′, the lengths of these perpendiculars AX and CY will be proportional to the sines of the angles of incidence and refraction NBA and N′BC, respectievly. Now it will be found that, no matter how we turn the disk, the perpendicular AX will always be about one-and-a-half times as long as the perpendicular CY. If we substitute for the half-disk of solid glass a hollow vessel of the same form and size with thin glass walls, and if we fill this vessel with water, we shall find now that the length of the perpendicular AX will always be about one-and-one-third times that of the perpendicular CY, because the relative index of refraction from air to water is 4/3, as above stated.

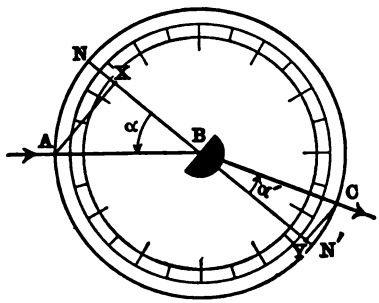


Fig. 43.—Optical Disk used to verify law of refraction.

But the best proof of the law of the refraction of light is to be found in the fact that this law is at the basis of the theory and construction of nearly all optical instruments, and it has been subjected, therefore, to the most searching tests. The law of refraction may also be regarded as completely verified by the methods that are employed in the determination of the indices of refraction of transparent bodies, solid, liquid and gaseous; which are described in

treatises on experimental optics usually under the title of "refractometry."

29. Reversibility of the Light Path.—When a ray of light AB is reflected at B in the direction BD, a plane mirror placed at D at right angles to BD will turn the reflected ray back on itself; arriving again at B, the light will obviously be reflected there so as to return finally to the point A where it started. This is a simple instance of a general law of optics known as the *principle of the reversibility of the light path*. Experiment shows that the same rule holds likewise in the case of the refraction of light, and that if ABC is the route pursued by light in going from a point A in one medium to a point C in an adjoining medium by way of the incidence-point B, and if then the light is reversed by some means so as to be started back along the path CB, it will be refracted at B into the first medium along the path BA. And, in general, if the final direction of the ray is reversed, for example, by falling normally on a plane mirror, the light will retrace its entire path, no matter how many reflections or refractions it may have suffered. Thus, in any optical diagram, in which the directions of the rays of light are indicated by arrow-heads, these pointers may all be reversed, if we wish to ascertain how the rays would go through the system if they were to enter it from the other end.

It follows, therefore, since

$$\frac{\sin \alpha}{\sin \alpha'} = n_{ab}, \qquad \frac{\sin \alpha'}{\sin \alpha} = n_{ba},$$

that we have the relation:

$$n_{ab} \cdot n_{ba} = 1;$$

that is, *the relative indices of refraction from* (a) *to* (b) *and from* (b) *to* (a) *are reciprocals of each other*. Thus, for example, since $n_{aw} = 4/3$ is the index from air to water, the index from water to air is $n_{wa} = 3/4$. Similarly, if $n_{ag} = 3/2$, the index from glass to air is $n_{ga} = 2/3$.

30. Limiting Values of the Index of Refraction.—Accordingly, we see that the value of the relative index of refraction may be greater or less than unity. If $n_{ab}>1$, the second medium (b) is said to be *more highly refracting* or (*optically*) *denser* than the first medium (a); and since in this case $\sin\alpha > \sin\alpha'$, it follows that $\alpha > \alpha'$, which means that *the refracted ray is bent towards the normal*, as happens when light is refracted from air to water ($n_{ab}=1.33$). On the other hand, if $n_{ab}<1$, the second medium (b) is said to be *less highly refracting* or (*optically*) *rarer* than the first medium (a), and now the angle of refraction (α') will be greater than the angle of incidence (α), so that in this case the refracted ray will be *bent away from the normal*, as, for example, when light is refracted from water into air ($n_{wa}=0.75$). Glass is more highly refracting than water, and diamond has the greatest light-bending power of all optical media, the index of refraction from air to diamond being about 2.5. The values of the constant n_{ab} for pairs of media a, b that are available for optical purposes are comprised within comparatively narrow limits, say, between 1/2 and 2. In the exceptional case when $n_{ab}=1$, the angles of incidence and refraction will be equal, and the rays pass from a to b without change of direction. This is the reason why a glass rod is invisible in oil of cedar. Sometimes accidental differences of refrangibility between two adjacent layers of the same medium enable us to distinguish one part of a transparent medium from another. Similarly, also, the presence of air-bubbles in water or glass is made manifest by the refractions that take place at the boundaries. A fish swimming in water does not see the water around him, but the phenomena of refraction may make him aware of the existence of a different medium above the surface of the water.

31. Huygens's Construction of a Plane Wave Refracted at a Plane Surface.—The straight lines AB and AD (Fig. 44) show the traces in the plane of the diagram of the plane wave-front advancing in the first medium (a) in the direc-

tion BD and the plane refracting surface, respectively. The disturbance is supposed to have just arrived at the point A of the refracting plane, which from this moment ($t=0$) becomes a new origin from which secondary hemispherical wavelets are propagated into the second medium (b). Now light is propagated with different velocities in different media; thus, for example, the velocity of light in water is only about three-fourths of what it is in air and the velocity in glass is about two-thirds of the velocity in air. Consequently, when waves of light pass from air into water or glass, the part of the wave-front that is in the denser medium advances more slowly than the part that is still in the air, so that the direction of the wave-front is changed in passing from one medium to another. Let the velocities of light in the media a and b be denoted by v_a and v_b, respectively. Then after a time $t=BD/v_a$, when the disturbance which was at B has just arrived at D on the boundary between the two media, the secondary wavelets which have been spreading out from A as center will have been propagated in the second medium (b) to a distance $AC=v_b.t=\frac{v_b}{v_a}.BD$; and, similarly, at the same instant from any intermediate point Q lying on AD between A and D the disturbance will have proceeded into the second medium (b) to a distance

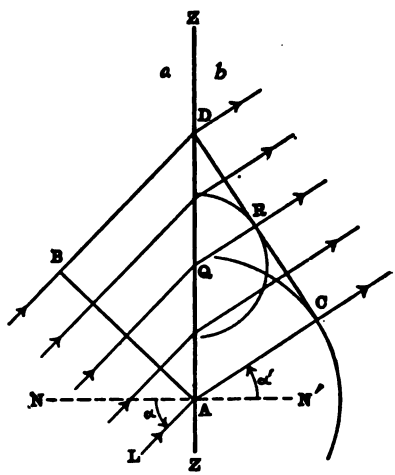


FIG. 44.—HUYGENS's construction of plane wave refracted at plane surface.

$$QR=\frac{v_b}{v_a}(BD-PQ)=\frac{v_b}{v_a}KD,$$

where K (not shown in the figure) designates the foot of the perpendicular let fall from Q on BD. Thus, the radii of the elementary cylindrical refracted waves whose axes are perpendicular to the plane of the diagram at A and Q are

$$\frac{v_b}{v_a}BD, \qquad \frac{v_b}{v_a}KD,$$

respectively; and, according to HUYGENS's principle, the refracted wave-front at this instant will be the surface which is tangent to all these elementary cylindrical surfaces. Exactly the same method as was used in the similar problem of reflection (§ 14) can be applied here; and thus it may be shown that at the moment when the disturbance reaches the point D of the plane refracting surface, the refracted wave-front will be the plane CD containing this point, which is perpendicular to the plane of the figure and tangent at C to the elementary wave represented by the spherical surface described about C as center with radius equal to AC. In the first medium the wave marches forward in the direction LA and in the second medium in the direction AC.

SNELL's law of refraction (§ 27) may be deduced from the figure by observing that $BD = AD.\sin\alpha$, where $\alpha = \angle NAL = \angle DAB$ denotes the angle of incidence, and $AC = AD.\sin\alpha'$, where $\alpha' = \angle N'AC = \angle ADC$ denotes the angle of refraction. Consequently,

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{BD}{AC} = \frac{v_a}{v_b} = \text{a constant},$$

which constant must, therefore, be identical with the relative index of refraction n_{ab}.

The diagram is drawn for the case when the light travels faster in the first medium than it does in the second ($v_a > v_b$), that is, when the second medium is more retarding or "optically denser" (§ 30) than the first.

32. Mechanical Illustration of the Refraction of a Plane Wave.—A simple mechanical illustration of the refraction of a plane wave at a plane surface may be devised as follows:

Two boxwood wheels each about two inches in diameter are connected by an iron axle about 4 inches long passing through the centers of the wheels at right angles to their planes of rotation (Fig. 45). If this body is placed on a smooth rectangular board, about a yard long and about 18 inches wide, which is slightly tilted, and allowed to roll diagonally down the board, its path will be along a straight line. But if a piece of felt cloth or velveteen cut in the form of a rectangle is glued in the middle of the board, with its long side parallel to the edge of the board, then when the body descends the inclined plane obliquely, one of the wheels will arrive at the edge of the cloth before the other, so that it will be suddenly slowed up while the other wheel continues to move on the bare board under the same conditions as before. Consequently, the axle will be made to swing round until both wheels get on to the cloth piece, the direction of motion having been abruptly changed in this process. At the opposite edge of the cloth rectangle, a similar change of the direction of motion takes place in an opposite sense, so that when the roller leaves the retarding surface and emerges again on to the bare board, it will be found to be going approximately in the same direction as at first. These bendings in the course of the roller descending the inclined plane at the places where it crosses the parallel sides of the cloth rectangle are analogous to the deviations in the line of march of a plane wave of light in traversing a glass slab surrounded by air.

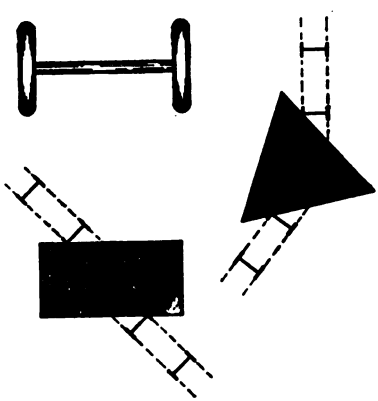


FIG. 45.—Mechanical illustration of refraction.

33. Absolute Index of Refraction.—If v_a, v_b and v_c denote the velocities of light in the media a, b and c, respectively, then, as we have just seen (§ 31), according to the wave-theory of light, the relative indices of refraction will be:

$$n_{ab}=\frac{v_a}{v_b},\ n_{ac}=\frac{v_a}{v_c},\ n_{bc}=\frac{v_b}{v_c};$$

and, hence, we find:

$$n_{ab}=\frac{n_{ac}}{n_{bc}};$$

so that in case we know the values n_{ac}, n_{bc} of the indices of a medium c with respect to each of the two media a and b, the value n_{ab} of the index of medium b with respect to medium a can be obtained at once by means of the above relation. Moreover, since (§ 29)

$$n_{cb}=\frac{1}{n_{bc}},$$

the preceding equation may be written as follows:

$$n_{ab}=n_{ac}.n_{cb}.$$

Thus, for example, suppose the three media a, b and c are water, glass and air, respectively; since $n_{ac}=3/4$ and $n_{cb}=3/2$, the index of refraction from water to glass is found by the above formula to be $n_{ab}=9/8$.

In fact, if there are a number of media $a, b, c, \ldots, i, j, k$ it is obvious that we shall have the following relation between the relative indices of refractions:

$$n_{ab}.n_{bc}\ \ldots\ n_{ij}.n_{jk}=n_{ak},$$

which is easily remembered by observing the order in which the letters occur in the subscripts. In particular, if the last medium k is identical with the first medium a, as is the case in an optical instrument surrounded by air, then $n_{ak}=n_{aa}=1$, and accordingly we obtain:

$$n_{ab}.n_{bc}\ \ldots\ n_{ij}.n_{ja}=1.$$

A special case of this general relation, viz.,

$$n_{ab}.n_{ba}=1,$$

has already been remarked (§ 29).

Since $n_{ac}.n_{ca}=n_{bc}.n_{cb}=1$ and $n_{ab}.n_{bc}=n_{ac}$, we may write also:

$$n_{ab}=\frac{n_{cb}}{n_{ca}},$$

and this formula suggests immediately the idea of employing some suitable medium c as a *standard optical medium* with respect to which the indices of refraction of all other media may be expressed. The natural medium to choose for this purpose is the ether itself which light traverses in coming to the earth from the sun and stars; and so the index of refraction of a medium with respect to empty space or vacuum is called its *absolute index of refraction* or simply its refractive index. Thus, the absolute index of refraction of vacuum (c) is equal to unity, that is, $n_c=1$. Similarly, the symbols n_a, n_b will be employed to denote the absolute indices of the media a, b, respectively; so that here they are really equivalent to the magnitudes denoted by n_{ca}, n_{cb} in the preceding formula, which, therefore, may be written:

$$n_{ab}=\frac{n_b}{n_a};$$

that is, *the relative index of refraction of medium* b *with respect to medium* a *is equal to the ratio of the absolute index of medium* b *to that of medium* a.

The absolute indices of refraction of all known transparent substances are greater than unity. The velocity of light in ordinary atmospheric air is so nearly equal to its velocity *in vacuo* that for all practical purposes we may generally take the absolute index of refraction of air as also equal to unity. The actual value for air at 0°C. and under a pressure of 76 cm. of mercury, for sodium light, is 1.000293.

With every isotropic medium there is associated, therefore, a certain numerical constant n called its (absolute) index of refraction; and, hence, when a ray of light is refracted from a medium of index n into another of index n',

the trigonometric formula for the law of refraction may be written thus:

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{n'}{n},$$

which may also be put in the following symmetric form:

$$n'.\sin\alpha' = n.\sin\alpha.$$

This latter mode of writing this relation suggests also another way of stating the fundamental fact in regard to the

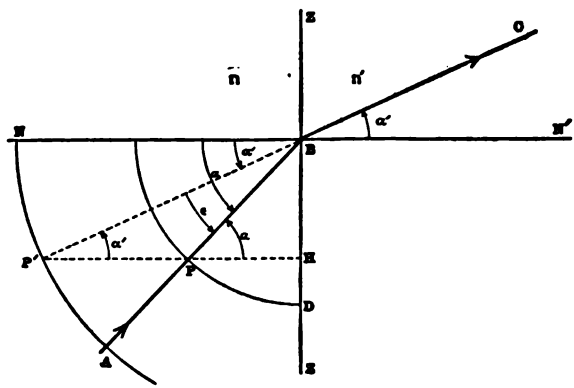


FIG. 46.—Construction of refracted ray ($n' > n$)

refraction of light, as follows: *Whenever a ray of light is refracted from one medium to another, the product of the index of refraction and the sine of the angle between the ray and the normal to the refracting surface has the same value after refraction* ($n'.\sin\alpha'$) *as before refraction* ($n.\sin\alpha$). This product $K = n.\sin\alpha = n'.\sin\alpha'$ which does not vary when the light crosses a surface separating a pair of isotropic media is called the *optical invariant of refraction.*

34. Construction of the Refracted Ray.—Let the absolute indices of refraction of two media separated from each other by a smooth refracting surface be denoted by n, n', and let the straight line AB (Figs. 46 and 47) represent the path in the first medium (n) of a ray incident on the boundary-

surface at the point B. The straight line NN′ represents the normal to the refracting surface at this point, and hence the plane of the diagram is the plane of incidence. The straight line ZZ shows the trace in this plane of the plane tangent to the refracting surface at the incidence-point B; in the special case when the refracting surface is itself plane, this straight line will be the trace of the surface of separation between the two media. With the point B as center

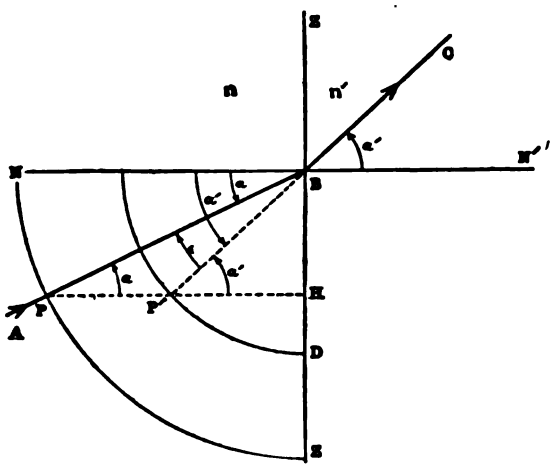


Fig. 47.—Construction of refracted ray ($n' < n$)

and with any radius r describe in the plane of incidence the arc of a circle cutting the incident ray AB in a point P lying in the first medium; and in the same plane, with radius n'/n times as great, that is, with radius $n'r/n$, describe also the arc of a concentric circle intersecting at P′ the straight line HP drawn through P perpendicular to ZZ at H. If the second medium is more highly refracting than the first, that is, if $n' > n$, the radius of the second circle will be greater than that of the first, as represented in Fig. 46; whereas when $n' < n$, the second circle is inside the first, as in Fig. 47. The path of the refracted ray correspodinng to the given

incident ray AB will be represented by the prolongation BC in the second medium of the straight line P'B.

The proof of this construction consists simply in showing that the $\angle N'BC$ between the normal and the straight line BC is equal to the angle of refraction α' as given by the formula $n'.\sin\alpha' = n.\sin\alpha$, where $\alpha = \angle NBA$ denotes the given angle of incidence. Evidently, from the figure, we have:

$$\frac{\sin\angle HPB}{\sin\angle HP'B} = \frac{BP'}{BP} = \frac{n'}{n},$$

and since $\angle HPB = \angle NBA = \alpha$, and $\angle HP'B = \angle N'BC$, we obtain immediately the relation: $n'.\sin\angle N'BC = n.\sin\alpha$ and therefore $\angle N'BC = \alpha'$.

35. Deviation of the Refracted Ray.—The acute angle through which the direction of the refracted ray has to be turned to bring it into the same direction as that of the incident ray is called the *angle of deviation* of the refracted ray and is denoted by ϵ; thus, $\epsilon = \angle P'BP$ (Figs. 46 and 47). Obviously,

$$\epsilon = \alpha - \alpha'.$$

The only ray incident at B whose direction will remain unchanged after the ray enters the second medium is the one that proceeds along the normal NB ($\alpha = \alpha' = \epsilon = 0$). The more obliquely the ray AB meets the refracting surface, that is, *the greater the angle of incidence, the greater also will be the deviation-angle.* The truth of this statement will be apparent from an inspection of the relation between the angles α and ϵ as exhibited in Fig. 46 or Fig. 47. The intercept PP' included between the circumferences of the two construction-circles, which remains constantly parallel to the incidence-normal, increases in length as the angle of incidence increases, whereas the other two sides BP, BP' of the triangle BPP', being always equal to the radii of the circles, remain constant in length; and hence the angle ϵ must increase in absolute value as the angle α increases.

36. Total Reflection.—In ordinary refraction, as we have seen, there can only be one refracted ray corresponding to

a given incident ray, but the question may be asked: Is it possible that, under certain circumstances, there will be no refracted ray, so that the incident light will be *totally reflected* at the surface without being refracted at all? Evidently such will be the case whenever in the foregoing construction (§ 34) the point P′ (Figs. 46 and 47) cannot be located, because the path of the refracted ray is determined by the straight line P′B.

Let us examine, first, the case when the second medium is more highly refracting than the first, $n'>n$ (Fig. 46).

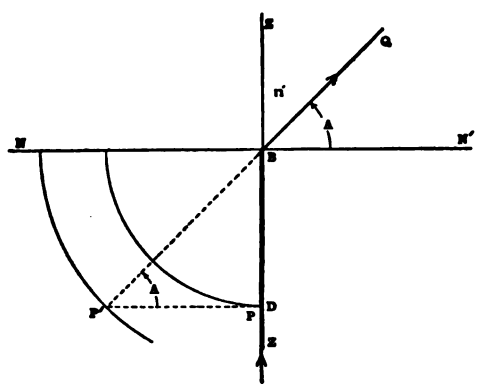


FIG. 48.—Limiting refracted ray ($n'>n$)

Suppose that the straight line AB which represents the path of the incident ray is initially in the position NB, and that it is rotated from this position around the point B as a pivot until it has turned through a right angle in the plane of the figure. While the point P on AB describes a quadrant of the circumference of the circle of radius BP, the point P′ will trace out an arc of the concentric circle of radius BP′, which, however, will never be equal to a quadrant of this circumference; for when the point P has completed its quadrant and arrived at the point D (Fig. 48) on the tangent plane drawn to the refracting surface at B, the point

P′ will likewise have reached the extremity of its arc where the tangent to the inner circle at D meets the circumference of the outer circle. The incident ray ZB just *grazes* the refracting surface at B or skims along it, and most of the light is reflected and does not enter the second medium at all, but the portion that is refracted pursues the path BQ corresponding to this extreme position of the point P′, and this will be the outermost of all the refracted rays that enter the second medium at the point B. The $\angle N'BQ = A$ which is the greatest value that the angle of refraction can have in the case when $n' > n$ is called the limiting or *critical angle* with respect to the two media. Since

$$\sin\angle N'BQ = \sin\angle PP'B = BD/BP' = n/n',$$

the magnitude of the angle A may be found from the relation:

$$\sin A = n/n', \quad (n < n');$$

which may likewise be derived by substituting the values $a = 90°$, $a' = A$ in the refraction-formula. Thus, if the first medium is air ($n = 1$) and the second medium is glass ($n' = 3/2$), $\sin A = 2/3$, so that the critical angle for air-glass is found to be $A = 41°\ 49'$. For air-water $\sin A = 3/4$, $A = 48°\ 35'$; and, consequently, a ray of light whose path lies partly in air and partly in water cannot possibly make an angle with the normal in the water greater than about $48°\ 30'$. For example, when a star is just rising or setting, the rays coming from it will fall very nearly horizontally on the surface of tranquil water and will be refracted into the water, therefore, at an angle of approximately $48°\ 30'$ with the vertical, so that if these rays entered an eye under the water, the star would appear to be nearly halfway to the zenith. In fact, all the rays coming into an eye placed under water from the entire overhanging arch of the sky would be comprised in the water within a cone whose axis points to the zenith and whose angular aperture is about 97°. In this connection it is interesting and instructive to examine a photograph of an air-scene made with a so-called

"fish-eye" camera immersed below the level of a clear pool of water, which affords some idea of how the world outside the pond must look to a fish. Professor Wood, of the Johns Hopkins University, has obtained a number of pictures of this kind, some of which are reproduced in illustrations in his very original book on *Physical Optics*, where also a brief description of the essential features of the ingenious pin-

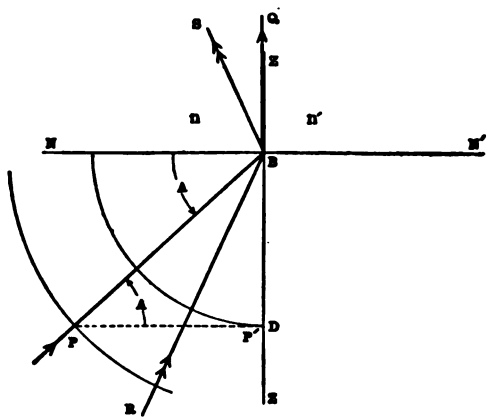


Fig. 49.—Limiting incident ray ($n' < n$)

hole camera which was used in making these pictures is also given.

Accordingly, when light is refracted from a rarer to a denser medium, there will always be a refracted ray corresponding to a given incident ray, because it is always possible under these circumstances to locate the position of the point P′ opposite P, or, to express it in another way, because when $n<n'$ there will always be a certain acute angle α' that will satisfy the equation $\sin\alpha' = n.\sin\alpha/n'$ for values of α comprised between 0° and 90°. But in the opposite case when the first medium is denser than the second ($n>n'$), for example, when the light is refracted from water to air, the statement just made is no longer true. The es-

sential difference in the two cases may be seen at once by reversing the arrow-heads in the diagram Fig. 48, at the same time making corresponding changes in the letters and symbols. Fig. 49 is a special diagram to illustrate this case. The refracted ray BQ which grazes the surface at the point B corresponds to the limiting incident ray PB which is incident at B at the critical angle $A = \angle NBP$; and, consequently, any ray, such as RB, which meets the surface at an angle of incidence greater than the angle A will be *totally reflected* in the direction BS. Thus, for values of a which are greater than the value A of the critical angle of incidence, there will be no value of a' that will satisfy the equation $\sin a' = n.\sin a/n'$ when $n > n'$. Only those rays incident at B which lie within the cone generated by the revolution of the limiting incident ray around the incidence-normal as axis will be refracted into the second medium; and all rays falling on the refracting surface at B and lying outside this cone will be totally reflected.

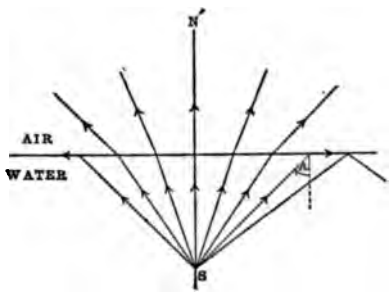


FIG. 50.—Refraction from water to air; total reflection.

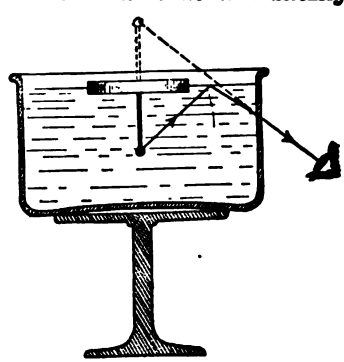


FIG. 51.—Experiment illustrating total reflection.

Fig. 50 shows how rays proceed from a radiant point below the horizontal free surface of still water.

If a pin is stuck in the under side of a flat circular cork floating on water, as represented in Fig. 51, and if the

diameter of the cork is (say) 6 inches and the head of the pin is not more than 2.5 inches below the water-level and vertically beneath the center of the cork, an eye placed anywhere above the level of the water will be unable to see the pin, because all the rays coming from it that meet the surface of the water beyond the edge of the cork will be totally reflected back into the water.

In Fig. 49 since $\sin \angle NBP = \sin \angle P'PB = BP'/BP$, we find in this case when $n' < n$ that $\sin A = n'/n$, which will also be obtained by putting $a = A$, $a' = 90°$ in the refraction-formula $n.\sin a = n'.\sin a'$. Comparing this result with the formula $\sin A = n/n'$ obtained for the case when $n' > n$, and recalling the fact that the sine of an angle is never greater than unity, we may formulate the following rule:

The sine of the so-called critical angle (A) *with respect to two media is the ratio of the index of refraction of the rarer to that of the denser medium.* Or, *the sine of the critical angle* (A) *of a substance is the reciprocal of the absolute index of refraction of the substance:* thus,

$$\sin A = \frac{1}{n}.$$

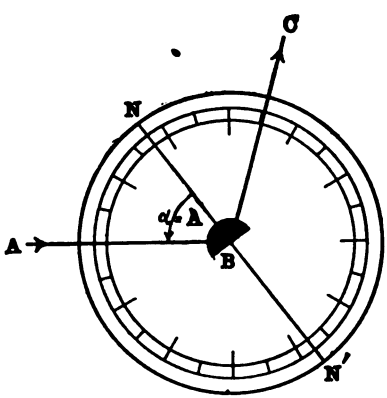


FIG. 52.—Optica Disk used to show total reflection.

37. Experimental Illustrations of Total Reflection.—The phenomenon of total reflection may be exhibited with the aid of the optical disk and the semicylinder of glass described in § 28. If the disk is turned so that the beam of incident parallel rays falls first on the curved surface of the semicylinder, as shown in Fig. 52, the rays meet this surface normally and proceed through the glass to the plane face without being deviated. At the plane surface a por-

tion of the beam is reflected and, in general, a portion is refracted from glass to air. If the disk is turned until the angle of incidence at the plane surface is just equal to the critical angle (A), the rays emerging into the air will proceed along the plane face, and if the disk is turned a little farther in the same sense, so that the angle of incidence exceeds the critical angle, the light will be totally reflected.

An ingenious contrivance for exhibiting the procedure of light in passing from water to air consists of a compara-

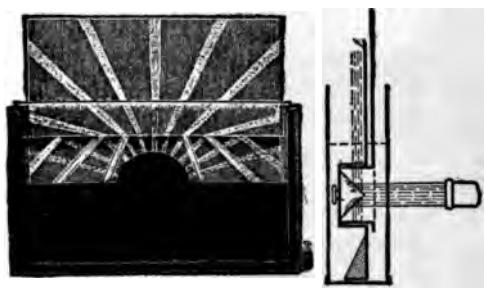


FIG. 53.—Demonstration of refraction from water to air and total reflection.

tively large glass tank (Fig. 53) filled with water and provided with a plane vertical metallic screen the lower half of which is under water while the upper half extends into the air above. A cylindrical beam of light is directed horizontally and normally against the lower part of the vertical glass wall of the tank, which is behind the screen and parallel to it. The rays entering the water are received first on the surface of a solid reflecting cone of aperture-angle 90° placed in the water under the screen and mostly in front of it, the axis of the cone being horizontal and its apex turned towards the on-coming light. From the surface of this cone the rays are reflected through the water in all directions in a vertical plane coinciding as nearly as possible with the front side of the screen turned towards the spectators. Surrounding the conical reflector and co-axial with

it, there is a cylindrical cavity of diameter very little larger than that of the base of the cone. The surface of this cylinder is made of thin sheet-metal blackened on the inside, wherein a number of equal horizontal slits are cut at equal angular distances apart, and through these slits narrow beams of light reflected from the surface of the cone are permitted to pass upwards towards the surface of the water, their courses being shown by the bright traces on the screen. Some of these beams will be refracted out into the air, whereas others, meeting the water-surface more obliquely, will be totally reflected.

If rays are incident normally on one of the two perpendicular faces of a glass prism (§ 48) whose principal section is an isosceles right-triangle (Fig. 54), they will enter the prism without deviation, and falling on the hypothenuse-face at an angle of 45°, which is greater than the critical angle of glass, they will be totally reflected there and turned through a right angle, so that they will emerge in a direction normal to the other of the two perpendicular faces of the prism.

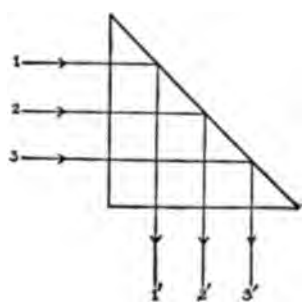


FIG. 54.—Total reflection prism

A prism of this kind is frequently employed in optical systems. It is used, for example, in connection with a photographic lens to rectify the image focused on the sensitive plate of the camera, so that the right and left sides of the negative will correspond to the right and left sides of the object. None of the light is lost by the total reflection in the prism, and if the prism is made of good optical glass of high transparency there will be comparatively little loss of light by absorption in the prism or by reflection on entering and leaving it. The same optical effect can be produced by a simple plane mirror, but as a rule a polished metallic surface absorbs the incident light to a considerable extent.

However, the loss of light in the case of a mirror silvered on glass is very slight; but on the other hand, the fine layer of silver may easily be injured mechanically or tarnished by exposure to the air. If the glass mirror is silvered on the back side, the light will be reflected from both surfaces of the glass and there will be confusion. Moreover, a glass mirror may easily get broken or become dislocated in an optical instrument; whereas a prism made of a solid piece of glass is much more substantial and durable.

Optical prisms consisting of solid pieces of highly transparent homogeneous glass with three or more polished plane faces are very extensively used in the construction of modern optical instruments for rectifying images which would otherwise be inverted or for bending the rays of light into new directions, etc. Usually the light undergoes several interior reflections before it issues from the prism, and these reflections are often total reflections. If the reflection is not total, it is best to silver the surface.

38. Generalization of the Laws of Reflection and Refraction. Principle of Least Time (Fermat's Law).—The laws of reflection and refraction, which merely describe the observed effects when light falls on the common surface of separation of two homogeneous media, and which are capable of simple explanation on the basis of the wave-theory, as has been illustrated in certain special cases (§§ 14 and 31), may be combined into a general law which was first announced about 1665 by the French philosopher FERMAT, and which may be stated as follows: *The actual path pursued by light in going from one point to another is the route that, under the given conditions, requires the least time.*

In case the reflections and refractions take place only at plane surfaces, the truth of the above statement is easily proved. Consider, first, the case *when the light is reflected from a plane mirror.* The straight line ZZ (Fig. 55) represents the trace of the plane mirror in the plane of the diagram, and A and C designate the positions of a pair of

points lying in this plane in front of the mirror. Now if a point X in the plane of the mirror is connected with A and C by the straight lines XA, XB, the route AXC will be shortest when the normal to the mirror at X lies in the plane AXC and bisects the angle AXC. The point X must lie, therefore, in the plane of the diagram at the point B, so that when AB is the direction of an incident ray, BC will be the direction of the reflected ray. Obviously, if A′ is the image of A in the mirror, then AB+BC=A′B+BC =A′C, and since the straight line A′C is shorter, for example, than (A′D+DC)=(AD+DC), where D is another point on the mirror different from the point B, it is evident that the route from A to C by way of B is shorter than the route *via* any other point on the mirror. Moreover, if the ray is reflected at a number of plane mirrors in succession, its entire path will be the shortest possible route from the starting point to the terminal point, subject to the condition that it must touch at each mirror in turn. The principle of least time in the case of reflection of light at a plane mirror dates back to the time of HERO of Alexandria (150 B. C.).

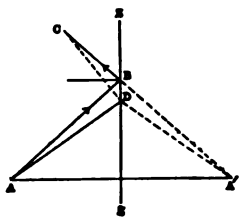


FIG. 55.—FERMAT'S principle of least time in case of reflection at a plane mirror.

When light is refracted at a plane surface, the route pursued between a point A in one medium to a point C in the other is indeed the quickest way but generally not the shortest. The following illustration will help to make the problem clear in this case. Suppose a level field is divided into two parts by a straight line ZZ (Fig. 56), on one side of which the ground is bare and smooth while on the other side it is plowed and rough; and let us also suppose that a man can walk only half as fast over the rough part of the field as over the smooth part, and that he desires to march as quickly as possible from a point A in the smooth ground

to a certain other point C in the plowed ground. The question is, Where should he cross the dividing line ZZ? Of course, his shortest route would be along the straight line from A to C which intersects ZZ at the point marked E in the figure, but unless the straight line AC happens to be perpendicular to ZZ this will not be his quickest way. Instead of crossing at E, suppose he selects a point F on ZZ which is a little nearer to his objective at C; then although the length FC in the plowed ground is shorter than before, on the other hand the distance AF over the smooth ground is longer, but on the whole we may assume that the route AFC will take less time than the shortest route AEC. But if the point of crossing ZZ is taken too far from E, the advantage of the shorter distance in the rough ground will presently be more than offset by the increasing length of the distance that has to be traversed in the smooth ground. Accordingly, there is a certain point B on ZZ such that the time taken along the route ABC will be the quickest of all routes. Now we shall see that this is also the very path that light would take if it were refracted from A to C across ZZ, supposing that the ratio of the velocities of light on the two sides of ZZ were the same as the ratio of the velocities of walking in the two parts of the field.

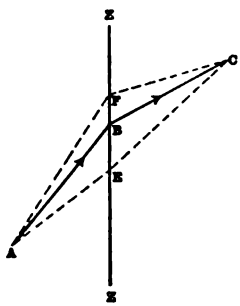


FIG. 56.—Quickest route from A to C *via* path ABC.

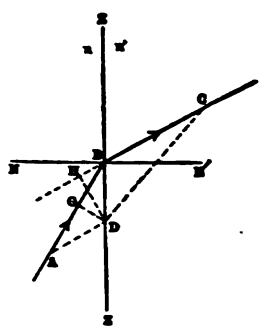


FIG. 57.—FERMAT'S principle of least time in case of refraction at plane surface.

In the accompanying diagram (Fig. 57) the broken line ABC represents the actual path of a ray of light from a

point A in the first medium (n) to a point C on the other side of the plane refracting surface ZZ in the second medium (n'); so that if NBN′ is the normal to the surface at B, then by the law of refraction:

$$\frac{\sin\angle \mathrm{NBA}}{\sin\angle \mathrm{N'BA}} = \frac{n'}{n} = \frac{v}{v'},$$

where v, v' denote the speeds with which light travels in the media n, n', respectively. The time taken to go over the route ABC is

$$t = \frac{\mathrm{AB}}{v} + \frac{\mathrm{BC}}{v'},$$

and we wish to show that this time t is less than the time

$$\frac{\mathrm{AD}}{v} + \frac{\mathrm{DC}}{v'}$$

along any other route ADC, where D designates the position of any point on ZZ different from the point B. Draw DG, DH perpendicular to AB, BC, respectively; then, since

$$\angle \mathrm{BDG} = \angle \mathrm{NBA}, \quad \angle \mathrm{BDH} = \angle \mathrm{N'BC},$$

evidently we have:

$$\frac{\sin\angle \mathrm{BDG}}{\sin\angle \mathrm{BDH}} = \frac{\mathrm{GB}}{\mathrm{HB}} = \frac{v}{v'}, \text{ or } \frac{\mathrm{GB}}{v} = \frac{\mathrm{HB}}{v'}.$$

Now

$$\frac{\mathrm{AB}}{v} + \frac{\mathrm{BC}}{v'} = \frac{\mathrm{AG}+\mathrm{GB}}{v} + \frac{\mathrm{BC}}{v'} = \frac{\mathrm{AG}}{v} + \frac{\mathrm{HC}}{v'};$$

and since AG<AD and HC<DC, therefore

$$\left(\frac{\mathrm{AB}}{v} + \frac{\mathrm{BC}}{v'}\right) < \left(\frac{\mathrm{AD}}{v} + \frac{\mathrm{DC}}{v'}\right),$$

and hence the time *viâ* ABC is less than it would be *viâ* any other route from A to C.

It should be remarked, however, that when the boundary-surface between two media is *curved*, the time taken by light to go from a point A across the surface to another point C is not always a minimum. It may, indeed, be a maximum, but it is always one or the other.

39. The Optical Length of the Light-path, and the Law of Malus.—In the time t that light takes to go along the path

ABC from a point A in one medium (n) to a point C in an adjacent medium (n') it would traverse *in vacuo* the distance

$$V.t = V\left(\frac{AB}{v} + \frac{BC}{v'}\right),$$

where V denotes the velocity of light *in vacuo*. But by the definition of the absolute index of refraction (§ 33), $n = V/v$, $n' = V/v'$; and hence the equivalent distance *in vacuo* is:

$$n.AB + n'.BC.$$

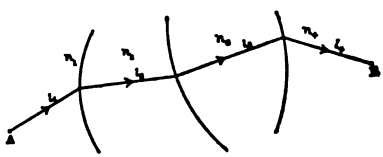


FIG. 58.—Optical length of ray-path $= \Sigma nl$.

The *optical length* of the path of a ray in a medium is defined to be the product of the actual length (l) of the ray-path by the index of the medium (n) that is, $n.l$. Suppose, for example, that light traverses a series of media n_1, n_2, etc., as represented in Fig. 58; the total optical length along a ray will be:

$$n_1.l_1 + n_2.l_2 + \ldots\ldots + n_m.l_m = \sum_{k=1}^{k=m} n_k.l_k;$$

where l_k denotes the actual length of the ray-path in the kth medium.

Now the wave-front at any instant due to a disturbance emanating from a point-source is the surface which contains all the farthest points to which the disturbance has been propagated at that instant. Thus, *the wave-surface* may be defined as *the totality of all those points which are reached in a given time by a disturbance originating at a point.* In a single isotropic medium the wave-surfaces, as we have seen, will be concentric spheres described around the point-source as center; but if the wave-front arrives at a reflecting or refracting surface μ, at which the directions of the so-called rays of light are changed, the form of the wave-surface thereafter will, in general, no longer be spherical; and even in those exceptional cases when the reflected or refracted wave-front is spherical, the waves will spread out from

a new center which is seldom identical with the original center. The function Σnl has the same value for all actual ray-paths between one position of the wave-surface and another position of it; so that when the form and position of the wave-front and the paths of the rays at any instant are known, the wave-front at any subsequent instant may be constructed by laying off equal optical lengths along the path of each ray.

A consequence of this definition of the wave-surface is that the ray is always normal to the wave-surface (§7), as will be evident from the following reasoning. Suppose that the straight line AB (Fig. 59) represents the path of a ray incident on the refracting surface ZZ at the point B, and that the straight line BC represents the path of the corresponding refracted ray. Moreover, let the wave-surface which passes through the point C be designated by σ. From the incidence-point B draw any other straight line, as BD, meeting the wave-surface σ in the point D. Then by the principle of least time, the route ABC is quicker, that is, optically shorter, than the route ABD, because the natural or actual route between the points A and D would not be by way of the incidence-point B. Hence, the straight line BC must be shorter than BD, and therefore BC is the shortest line that can be drawn from the incidence-point B to the wave-surface σ.

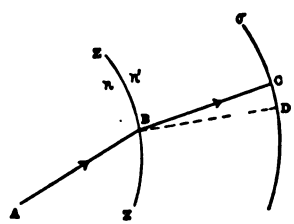


FIG. 59.—Law of MALUS: Ray normal to wave-front.

The same reasoning is applicable to all cases of reflection and refraction, and hence we may make the following general statement:

Rays of light meet the wave-surface normally; and, conversely, *The system of surfaces which intersect at right angles rays emanating originally from a point-source is a system of wave-surfaces.*

This law was published by MALUS in 1808.

PROBLEMS

1. (*a*) A ray is refracted from vacuum into a medium whose index of refraction is $\sqrt{2}$, the angle of incidence being 45°: find the angle of refraction.

(*b*) Find the angle of incidence of a ray which is refracted at an angle of 30° from vacuum into a medium of index equal to $\sqrt{3}$.

(*c*) Find the relative index of refraction when the angles of incidence and refraction are 30° and 60°, respectively. Ans. (*a*) 30°; (*b*) 60°; (*c*) $\sqrt{3}$: 3.

2. Assuming that the indices of refraction of air, water, glass and diamond have the values 1, $\frac{4}{3}$, $\frac{3}{2}$ and $\frac{5}{2}$, respectively, calculate the angle of refraction in each of the following cases:

(*a*) Refraction from air to glass, angle of incidence 40°; (*b*) from air to water, angle of incidence 60°; (*c*) from air to diamond, angle of incidence 75°; (*d*) from glass to water, angle of incidence 30°; (*e*) from diamond to glass, angle of incidence 36° 52′ 11.6″. Ans. (*a*) 25° 22′ 26″; (*b*) 40° 30′ 19″; (*c*) 22° 43′ 44″; (*d*) 34° 13′ 44″; (*e*) 90°.

3. The height of a cylindrical cup is 4 inches and its diameter is 3 inches. A person looking over the rim can just see a point on the opposite side 2.25 inches below the rim. But when the cup is filled with water, looking in the same direction as before, he can just see the point of the base farthest from him. Find the index of refraction of water. Ans. 4:3.

4. The index of a refracting sphere is $\sqrt{3}$; it is surrounded by air. A ray of light, entering the sphere at an angle of incidence of 60° and passing over to the other side, is there partly reflected and partly refracted. Show that the reflected ray and the emergent ray are at right angles to each other.

5. In the preceding problem, show that the reflected ray will cross the sphere again and be refracted back into the

air in a direction exactly opposite to that which the ray had before it entered the sphere.

6. A straight line drawn through the center C of a spherical refracting surface meets the surface in a point designated by A. If J, J′ designate the points where an incident ray and the corresponding refracted ray intersect the straight line AC, and if $CJ=\frac{n'}{n}.AC$, show that $CJ'=\frac{n}{n'}.AC$, where n, n' denote the indices of refraction of the first and second media, respectively.

7. Construct the path of a ray refracted at a plane surface. Draw diagrams for the cases when n' is greater and less than n. Construct the critical angle in each figure.

8. The velocity of light in air is approximately 186000 miles per second. How fast does it travel in alcohol of index 1.363? Ans. Approximately, 136 460 miles per sec.

9. A fish is 8 feet below the surface of a pool of clear water. A man shooting at the place where the fish appears to be points his gun at an angle of 45°. Where will the bullet cross the vertical line that passes through the fish? (Take index of water as 1.33, and neglect any deflection of the bullet caused by impact with the water.)

Ans. 3 feet above the fish.

10. Assuming that the velocity of light in air is 30 000 000 000 cm. per sec., calculate its velocity in water and in glass.

11. Prove that $n_{ab}=n_{cb}:n_{ac}$.

12. Show that the sine of the critical angle of an optical medium is equal to the reciprocal of the absolute index of refraction.

13. Assuming same values of the indices of refraction as in problem No. 2, calculate the values of the critical angle for each of the following pairs of media: (*a*) air and glass, (*b*) air and water, (*c*) air and diamond.

Ans. (*a*) 41° 48′ 40″; (*b*) 48° 35′ 25″; (*c*) 23° 34′ 41″.

14. A 45° prism is used to turn a beam of light by total internal reflection through a right angle. What must be

the least possible value of the index of refraction of the glass? Ans. $\sqrt{2}$.

15. Show that when a ray of light passes from air into a medium whose index of refraction is equal to $\sqrt{2}$, the deviation cannot be greater than 45°.

16. The absolute index of refraction of a certain transparent substance is $\frac{5}{3}$. Show that a luminous point at the center of a cube of this material cannot be seen by an eye in the air outside, if at the center of each face of the cube a circular piece of opaque paper is pasted whose radius is equal to three-eighths of the edge of the cube.

17. What will be the greatest apparent zenith distance of a star to an eye under water?

18. Explain why it is that it is not possible for a person by merely opening his eyes under water to see distinctly objects in the water around him or in the air above the water; whereas, if he is provided with a diver's helmet with a plate glass window in it, he will experience no difficulty in distinguishing such objects clearly.

19. Rays of light are emitted upwards in all directions from a luminous point at the bottom of a trough containing a layer of a transparent liquid 3 inches in depth and of refractive index 1.25. Show that all rays which meet the surface outside a certain circle whose center is vertically above the point will be totally reflected; and find the radius of this circle. Ans. 4 inches.

20. A pin with a white head is stuck perpendicularly in the center of one side of a flat circular cork, and the cork is floated on water with the pin downwards. Assuming that the head of the pin is 2 inches below the surface of the water, find the smallest diameter the cork can have so that a person looking down through the water (index $\frac{4}{3}$) from the air above (index unity) could not see the head of the pin. Ans. 4.535 inches.

21. Plot a curve showing the deviation ϵ as a function of the angle of incidence α for the case when the refraction is from water ($n = 4/3$) to air ($n' = 1$).

CHAPTER IV

REFRACTION AT A PLANE SURFACE, AND ALSO THROUGH A PLATE WITH PLANE PARALLEL FACES

40. Trigonometric Calculation of Ray Refracted at a Plane Surface.—A geometrical construction of the path of the refracted ray was given in § 34. The path of a ray refracted at a plane surface may also be easily determined by trigonometric calculation. The straight line *yy* in Figs. 60 and 61 represents the plane refracting surface separating the two media of indices n, n', and the straight line LB shows the path of a ray which is incident on *yy* at the point marked B. The straight line LA perpendicular to *yy* at A is the *axis* of the refracting plane with respect to the position of the point L. The magnitudes $v = AL$ $\alpha = \angle ALB$ which determine completely the position of the incident ray are sometimes called the *ray-coördinates*. Let L′ designate the point where the refracted ray L′B intersects the axis *xx*, and let $v' = AL'$, $\alpha' = \angle AL'B$ denote the coördinates of the refracted ray. The

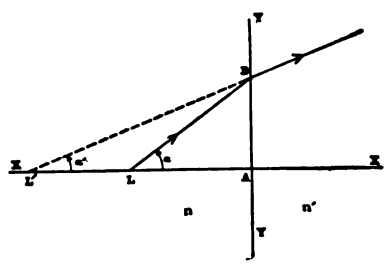


FIG. 60.—Refraction of ray at plane surface: $v = AL$, $v' = AL'$ $(n' > n)$.

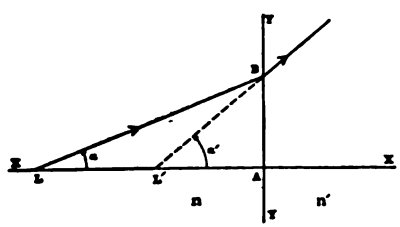


FIG. 61.—Refraction of ray at plane surface: $v = AL$, $v' = AL'$ $(n' < n)$.

problem is: Given the incident ray (v, a), determine the refracted ray (v', a').

From either diagram we obtain immediately the relation:

$$\frac{v'}{v} = \frac{\tan a}{\tan a''};$$

and since $n.\sin a = n'.\sin a'$, we obtain finally the following formulæ for calculating the refracted ray:

$$v' = \frac{v}{n} \frac{\sqrt{n'^2 - n^2.\sin^2 a}}{\cos a}, \sin a' = \frac{n}{n'}.\sin a.$$

Now if the point L is a luminous point, rays will emanate from it in all directions, and, whereas the magnitude v will remain the same for all these rays, the angle a will vary from ray to ray. But for different values of a, in general we shall obtain different values of the magnitude v', and, consequently, the position of the point L′ on the axis will be different for different incident rays coming from L. Accordingly, the bundle of refracted rays corresponding to a homocentric bundle of incident rays will not be homocentric.

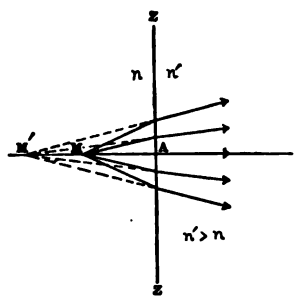


FIG. 62.—Refraction of paraxial rays at plane surface: $u = AM$, $u' = AM'$, $u' : n' = u : n$, $(n' > n)$.

41. Imagery in a Plane Refracting Surface by Rays which Meet the Surface Nearly Normally.—The more or less blurred and distorted appearance of objects seen under water is familiar to everybody. When the rays that enter the eye meet the surface of the water very obliquely, the distortion is almost grotesque. If the pupil of the eye were not comparatively small, it would indeed be practically almost impossible to recognize an object under water, even if the eye were placed in the most favorable position vertically over the object. It is only because the apertures of the bundles of effective rays that enter the eye

are quite narrow, that there is any true image-effect at all in the case of refraction at a plane surface.

When the eye looks directly along the normal to the plane refracting surface at an object-point M on the other side of the surface (Figs. 62 and 63), the effective rays coming from M will meet the surface very nearly perpendicularly, and the incidence-points will all be so close to the point A that there will be practically no difference between the lengths of the straight lines MA and MB, and accordingly under these circumstances we may write sin α in place of tan α. Similarly, also, with respect to the refracted ray, sin α' can be substituted here for tan α'. And if in this case we put $AM = u$, $AM' = u'$, where M, M' designate the points where a ray which is very nearly normal to the refracting plane crosses the normal before and after refraction, we have therefore,

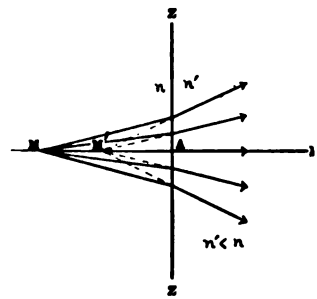


FIG. 63.—Refraction of paraxial rays at plane surface: $u = AM$, $u' = AM'$, $u' : n' = u : n$, $(n' < n)$.

$$\frac{u'}{u} = \frac{\tan \alpha}{\tan \alpha'} = \frac{\sin \alpha}{\sin \alpha'},$$

and, hence, by the law of refraction:

$$\frac{n'}{u'} = \frac{n}{u}, \quad \text{or} \quad u' = \frac{n'}{n} . u.$$

The angle α has disappeared entirely from this formula, and the value of u' may be found as soon as the value of u is given. This means that corresponding to a given position of the object-point M there is a perfectly definite image-point M', and the points M, M' are said to be a pair of *conjugate points*. Accordingly, *when a narrow bundle of homocentric rays is incident nearly normally on a refracting plane, the corresponding bundle of refracted rays will be homocentric also.* And if

the aperture of the bundle is infinitely narrow, the imagery will be ideal.

For example, a pebble at the bottom of a pool of water 12 inches deep will be seen distinctly from a point in the air vertically above it, but it will appear to be only 9 inches below the surface of the water, since $n'/n = 3/4$. On the other hand, an object 9 inches above the surface will seem to be 12 inches above it to an eye in the water vertically beneath the object, because in this case $n'/n = 4/3$.

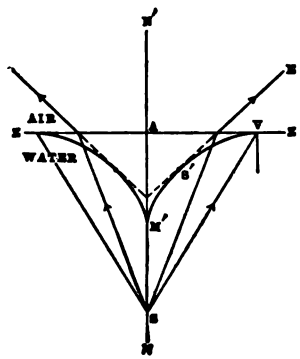


FIG. 64.—Caustic by refraction at plane surface from water to air.

42. Image of a Point Formed by Rays that are Obliquely Refracted at a Plane Surface.—But if the bundle of rays coming from the luminous point S (Fig. 64) is a wide-angle bundle of considerable aperture, no distinct image will be formed by these rays after refraction at a plane, but the points of intersection of the refracted rays will be spread over a so-called *caustic surface*, which in this case is a surface of revolution around the normal SA drawn from S to the refracting plane. The figure shows a meridian section of this surface for the case when the rays are refracted from a denser to a rarer medium ($n' < n$), the curve in this case being the evolute of an ellipse. Each refracted ray produced backwards touches the caustic surface. The cusp of the meridian curve is on the normal SA at the point M′ where the image of S is formed by rays that meet the refracting plane nearly perpendicularly, as explained in the preceding section. Wherever the eye is placed in the second medium, only a narrow bundle of rays coming from S can enter it through the pupil of the eye. The nearest approach to an image of the source at S as seen by rays that are refracted more or less obliquely

will be the little element of the caustic surface which is the assemblage of the points where the effective rays that enter the eye touch this surface. Thus, rays entering the eye at E appear to come from the point S′ where the tangent from E touches the caustic. It is evident now why an object S under

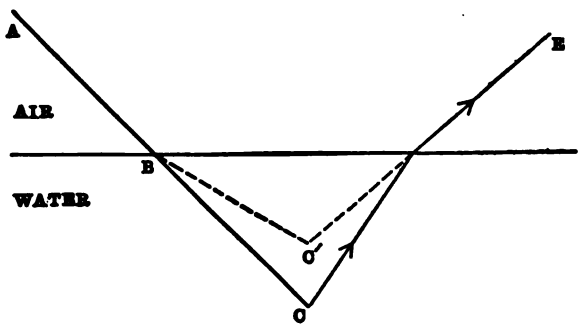


Fig. 65.—Rod partly immersed in water appears to be bent upwards.

water appears to be raised towards the surface and at the same time also to be shifted towards the spectator more and more as the eye at E is brought nearer to the surface of the water, until finally when the eye is on a level with the surface of the water, the image of S appears now to be at V on the refracting plane. Rays from S that meet the surface beyond this limiting point V where the caustic curve is tangent to the straight line ZZ will be totally reflected. The image of S seen by the eye at E is blurred and distorted, because the image-point S′ is the point of intersection of a very limited portion of the bundle of refracted rays that enter the eye.

The above explanation makes it clear why a straight line ABC (Fig. 65) which is partly in air and partly in water will appear to an eye at E to be bent at B into the broken line ABC′. The image BC′ of the part BC under water can be plotted point by point for any position of the eye.

43. The Image-lines of a Narrow Bundle of Rays Refracted Obliquely at a Plane.—The diagram (Fig. 66) shows the paths of two rays SBD and SCE which originating at S and falling on the refracting plane ZZ at the points B and C are refracted in the directions CE and BD into the eye of an observer. The refracted rays produced backwards intersect at S′ and cross the normal SA at the points marked W and V. Evidently, all the rays from S that fall on the refracting plane at points between B and C will, after refraction, intersect SA at points between V and W. Suppose that the figure is revolved around SA as axis, then each ray will generate a conical surface, and the vertices of these cones will be at the points S, V, and W for the rays that are actually drawn in the diagram. The bundle of rays that enter the eye at DE will be a small portion of the refracted rays that are contained between the conical surfaces whose vertices are at V and W. These conical surfaces intersect each other in the circle which is described by the point S′ when the figure is rotated around the axis SA, and it is a little element of arc of this circle perpendicular to the plane of the diagram at S′ that contains the points of intersection of the rays that enter the eye. This is called the *primary image-line* (§188) of the narrow bundle of refracted rays. There is another image line at V called the *secondary image-line*, which lies in the plane of the paper, and which is generally taken as perpendicular to the axis of the bundle of refracted rays, though sometimes it is considered as the segment VW of the axis of revolution. But these are intricate matters that can be only alluded to in this place. (See Chapter XV.)

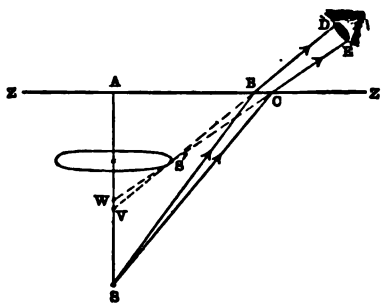


FIG. 66.—Oblique refraction at plane surface ($n' < n$).

44. Path of a Ray Refracted Through a Slab with Plane Parallel Sides.—When a ray of light traverses several media in succession, then

$$n_1 . \sin \alpha_1 = n_2 . \sin \alpha_1', \quad n_2 . \sin \alpha_2 = n_3 . \sin \alpha_2', \text{ etc.},$$

where n_1, n_2, n_3, etc., denote the indices of refraction of the media, and α_1, α_1'; α_2, α_2'; etc., denote the angles of incidence and refraction at the various surfaces of separation. In the special case when these refracting surfaces are a *series of parallel planes*, the angle of incidence at one plane will be equal to the angle of refraction at the preceding plane ($\alpha_{k+1} = \alpha_k'$, where the integer k denotes the number of the plane).

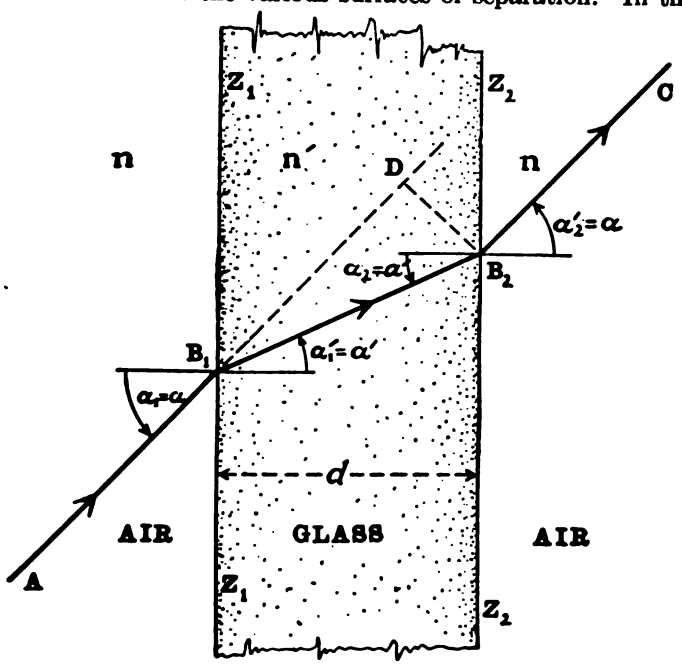


FIG. 67.—Path of ray refracted through plate with plane parallel sides.

The simplest case of this kind occurs when there are only *two parallel refracting planes*, and when the last medium is

the same as the first, as, for example, in the case of a slab of glass bounded by plane parallel sides and surrounded by air, as represented in Fig. 67. Then

$$n_3 = n_1 = n,\ n_2 = n',$$

and

$$a_1' = a_2 = a'.$$

Accordingly, we have the following pair of equations:

$$n.\sin a_1 = n'.\sin a',\ n'.\sin a' = n.\sin a_2';$$

and, therefore:

$$a_2' = a_1 = a;$$

which means that the ray emerges from the slab in the same direction as it entered it. Thus, *when a ray of light traverses a slab with plane parallel sides which is bounded by the same medium on both sides, the emergent ray will be parallel to the incident ray.* Obviously, this statement may be amplified as follows: *When a ray of light traverses a series of media each separated from the next by one of a series of parallel refracting planes, the final and original directions of the ray will be parallel, provided the first and last media have the same index of refraction.*

The only effect of the interposition of the glass plate (Fig. 67) in the path of the ray is to shift the path to one side without altering the direction of the ray. It might be inferred, therefore, that the apparent position of an object as seen through such a plate of glass would not be altered, but this is not true in general, as we shall proceed to explain. Every ray that traverses the plate will be found to be displaced at right angles to its original position through a distance

$$B_2D = \frac{\sin(a - a')}{\cos a'} d,$$

where d denotes the thickness of the plate. Since

$$\cos a' = \frac{\sqrt{n'^2 - n^2.\sin^2 a}}{n'},$$

the formula above may be put also in the following form:

$$B_2D = \frac{\sin a\,(\sqrt{n'^2 - n^2 \sin^2 a} - n.\cos a)}{\sqrt{n'^2 - n^2.\sin^2 a}}.d.$$

Accordingly, the shift B_2D varies with the slope of the incident ray. If the object is very far away, the rays that enter the eye will be parallel, so that the apparent position of a distant object will not be altered in the slightest by viewing it through a plate of glass with plane parallel sides, no matter what may be the angle of incidence of the rays, and consequently the plate may be turned to the rays at dif-

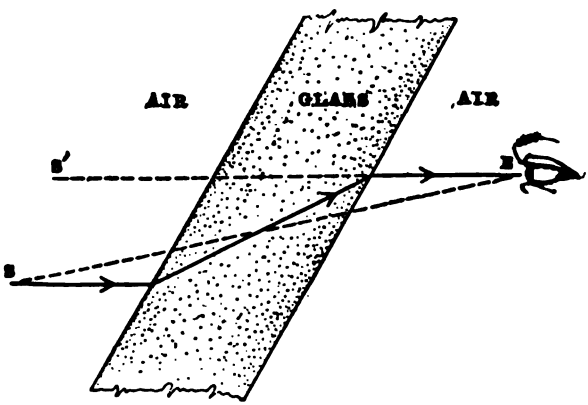


FIG. 68.—Apparent position of object seen through plate with plane parallel sides

ferent angles without producing any change in the appearance of the object as seen through it. But if the object-point S (Fig. 68) is near at hand, an eye at E will see it in the direction ES, but when the glass is interposed, it will appear to lie in the direction FS′ which is sensibly different from ES, and this difference can be increased or diminished by rotating the plate around an axis perpendicular to the plane of the figure. This principle is utilized very ingeniously in the original form of ophthalmometer designed by HELMHOLTZ (1821–1894) for measuring the curvatures of the refracting surfaces of the eye. It is employed also in an instrument for measuring the diameter of a microscopic object, which Professor POYNTING has called the "parallel plate micrometer" (see *Proc. Opt. Convention*, London, 1905, p. 79).

45. Segments of a Straight Line.—The finite portion of a straight line included between two points is called a *segment* of the line, while each of the other two parts of the line is to be regarded as a prolongation of the segment. Considered as generated by the motion of a point along a straight line from a starting-point or origin A to an end-point or terminus B, the segment AB is frequently spoken of also as the *step* from A to B or the step AB. The order of naming the two capital letters placed at the ends of a segment describes the sense of the motion or the direction of the segment. Thus, with respect to direction the step BA (Fig. 69) is exactly the reverse of the step AB.

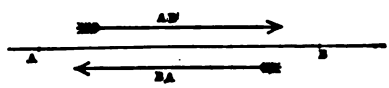


FIG. 69.—Segments of a straight line: AB = −BA.

Two steps AB and CD are said to be congruent, that is,

$$AB = CD,$$

provided these steps are not only equal in length but executed in the same sense.

If A, B, C are three points ranged along a straight line in any order, that is, if AB and CD are two steps along the same straight line such that the end of one step is the starting point of the other, then the step AC is said to be equal to the sum of the steps AB and BC; thus,

$$AB + BC = AC;$$

and hence also:

$$AB = AC - BC, \quad BC = AC - AB.$$

Moreover, if we suppose that the point C is identical with the point A, it follows that

$$AB + BA = 0 \text{ or } AB = -BA.$$

Thus, if one of the two directions along a straight line is regarded as the *positive direction*, the opposite direction is to be reckoned as *negative*. For example, if the distance between A and B is equal to 12 linear units, and if we put $AB = +12$, then $BA = -12$.

Similarly, also, we may write:

$$AB+BC+CA=0;$$

or if X designates the position of any fourth point on the straight line, then

$$AB+BC+CX=AX.$$

These ideas will be found to be of great service in treating a certain class of problems in geometrical optics; and an application of this method of adding line-segments occurs in the following section.

46. Apparent Position of an object seen through a transparent Slab whose Parallel Sides are perpendicular to the Line of Sight.—In Fig. 70 the line of sight joining the object-point M_1 with the spectator's eye at E is perpendicular at A_1 and A_2 to the parallel faces of the transparent slab, and all the rays that enter the eye will pass through the slab close to this axial line. Inside the slab they will proceed as if they had originated at a point M_1' on the line of sight, but being again refracted, they will emerge into the surrounding medium as if they had come from a point M_2', which is the apparent position of the object-point as seen by rays that are very nearly perpendicular to the faces of the slab. If n, n' denote the indices of refraction of the two media, then, according to §§ 41 and 45, we may write the following equations:

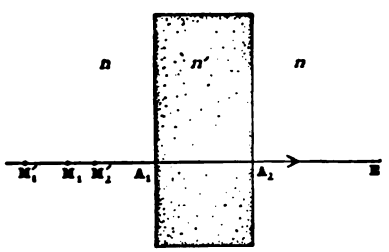


FIG. 70.—Displacement of object viewed perpendicularly through plate with plane parallel sides.

$$\frac{A_1M_1}{n}=\frac{A_1M_1'}{n'},\quad A_2M_1'=A_2A_1+A_1M_1',\quad \frac{A_2M_1'}{n'}=\frac{A_2M_2'}{n}.$$

Hence, the apparent displacement of the object is:

$$\begin{aligned} M_1M_2' &= M_1A_1 + A_1A_2 + A_2M_2' \\ &= M_1A_1 + A_1A_2 + \frac{n}{n'}A_2M_1' \\ &= M_1A_1 + A_1A_2 + \frac{n}{n'}(A_2A_1 + A_1M_1') \\ &= M_1A_1 + A_1A_2\left(1 - \frac{n}{n'}\right) + A_1M_1 = \frac{n'-n}{n'}A_1A_2; \end{aligned}$$

accordingly, if the thickness of the plate is denoted by $d = A_1A_2$,

$$M_1M_2' = \frac{n'-n}{n'}d.$$

Thus, we see that the apparent displacement in the line of sight depends only on the thickness of the plate and on the relative index of refraction ($n' : n$), and is entirely independent of the distance of the object-point from the slab. Hence, also, the size of the image of a small object viewed directly through a glass plate is the same as that of the object, but its apparent size will be different, because since the image and object are at different distances from the eye, the angles which they subtend will be different.

An object viewed perpendicularly through a glass plate surrounded by air ($n' : n = 3 : 2$) will appear to be one-third the thickness of the plate nearer the eye than it really is.

If the displacement of the object is denoted by x, that is, if we put $M_1M'_2 = x$, then

$$\frac{n'}{n} = \frac{d}{d-x}.$$

This relation has been utilized in a method of determining the relative index of refraction ($n' : n$). A microscope S pointed vertically downwards is focused on a fine scratch or object-point O. A plate of the material whose index is to be determined is then inserted horizontally between the object and the objective of the microscope. The interposition of the plate necessitates a re-focusing of the microscope in order to see the object distinctly, which will

now appear to be at a point O′ nearer the microscope by the distance $x = OO'$. This distance x is easily ascertained in terms of the distance through which the objective of the microscope has to be raised in order to obtain a distinct image of the object. The thickness of the plate is easily measured, and, consequently, we have all the data for determining the value of n'/n. This method is especially convenient for obtaining the index of refraction of a liquid (Fig. 71).

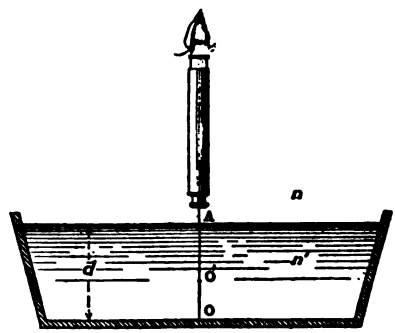


Fig. 71.—Measurement of index of refraction of a liquid.

47. Multiple Images in the two Parallel Faces of a plate glass Mirror.—An object is reproduced in a metallic mirror by a single image, but in a glass mirror which is silvered on the back side there will be a series of images of an object in front of the glass, which may be readily seen by looking a little obliquely at the reflection of a candle-flame in an ordinary looking glass. The first image will be comparatively faint, the second one the brightest and most distinct of all, and behind these two principal images other images more or less shadowy may also be discerned whose intensities diminish rapidly until they fade from view entirely. These multiple images by reflection may also be seen in a transparent block of glass with plane parallel sides.

The light falling on the first surface is partly reflected and partly refracted. It is this reflected portion that gives rise to the first image of the series. The rays that are refracted across the plate will be partly reflected at the second face, and, returning to the first face, a portion of this light will be refracted back into the air and give rise to the second image of the series; while the other portion of the light will be re-

flected back into the glass to be again reflected at the l face, and so on. In the diagram (Fig. 72) the source of light is supposed to be at the point marked S, and the stra

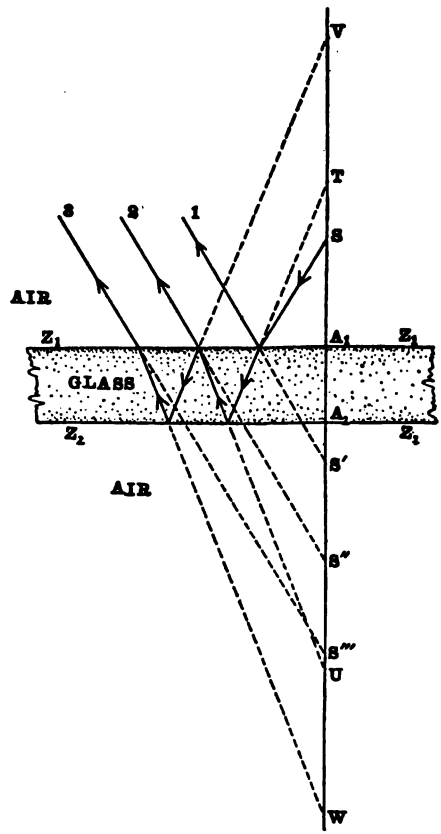


FIG. 72.—Multiple images by reflection from the two parallel faces of a plate of glass.

line drawn from S perpendicular to the parallel faces of glass slab meets these faces in the points marked A_1 and The path of one of the rays coming from S is indicate

the figure, and it can be seen how it zigzags back and forth between the two sides of the slab, becoming feebler and feebler in intensity at each reflection. We consider here only such rays from S as meet the surface very nearly normally. The series of images of S will be formed at S′, S″, S‴, etc., all lying on the prolongation of the normal SA_1A_2, and it is because these images are all ranged in a row one behind the other, that ordinarily when we look in a mirror we do not see the images separated.

The reflected ray 1 proceeds as if it had come from S′, the position of this point being determined by the relation $A_1S' = SA_1$. But the refracted ray crosses the slab as if it had come from the point T, the position of which is determined by the relation $TA_1 = n.SA_1$, where n denotes the index of refraction of the glass (the other medium being assumed to be air of index unity). Arriving at the second face, this ray will be reflected as if it had come from a point U such that $A_2U = TA_2$. Returning to the first surface, it will be partly refracted out into the air as the ray marked 2 proceeding as if it came from the second image-point S″, the position of which is determined by the relation $A_1S'' = A_1U/n$; and also partly reflected as if it had come from a point V such that $VA_1 = A_1U$. The ray is reflected a second time at the second face, as if it came from the point W, where $A_2W = VA_2$; and being once more refracted at the first face, emerges into the air as the ray marked 3, appearing now to come from the image-point marked S‴ determined by the relation $A_1S''' = A_1W/n$.

What is the interval between one image and the next? For example, let us try to obtain an expression for the interval S″S‴. This may be done as follows:

$$S''S''' = S''A_1 + A_1S''';$$

$$\begin{aligned} A_1S''' &= A_1W/n = (A_1A_2 + A_2W)/n = (A_1A_2 + VA_2)/n \\ &= (A_1A_2 + VA_1 + A_1A_2)/n = (A_1U + 2A_1A_2)/n \\ &= A_1S'' + 2d/n; \end{aligned}$$

where $d = A_1A_2$ denotes the thickness of the glass plate. Hence, we find:

$$S''S''' = \frac{2d}{n}.$$

It appears, therefore, that the distance between one image and the next is constant and equal to $\frac{2}{n}$ times the thickness of the plate. Thus, for a glass plate for which $n = 3/2$ the distance from one image to the next is equal to 4/3 the thickness of the plate.

PROBLEMS

1. A ray of light traverses in succession a series of isotropic media bounded by parallel planes, and emerges finally into a medium with the same index of refraction as that of the first medium. Show that the final path of the ray is parallel to its original direction.

2. Construct accurately the paths of six rays proceeding from a point below the horizontal surface of water and refracted into air; and show where the object-point will appear to be as seen by an eye above the surface of the water, for three different positions of the eye.

3. Why does the part of a stick obliquely immersed in water appear to be bent up towards the surface of the water? Explain clearly.

4. Derive the formula $\frac{n'}{u'} = \frac{n}{u}$ for the refraction of paraxial rays (§63) at a plane surface.

5. A ray of light incident on a plane refracting surface at an angle α crosses a straight line drawn perpendicular to the surface at a distance v from this surface. How far from the surface does the refracted ray cross this line?

6. If a bird is 36 feet above the surface of a pond, how high does it look to a diver who is under the water? What is the apparent depth of a pool of water 8 feet deep?

Ans. 48 feet above the surface; 6 feet.

7. What will be the effect on the apparent distance of an object if a slab of transparent material with plane parallel sides is interposed at right angles to the line of vision?

Ans. It will appear to be nearer the eye by the amount $(n-1)/d$, where d denotes the thickness of the slab and n denotes the index of refraction of the material.

8. A cube of glass of index of refraction 1.6 is placed on a flat, horizontal picture; where does the picture appear to be to an eye looking perpendicularly down on it?

Ans. It will appear to be raised three-eighths of the thickness of the cube.

9. A microscope is placed vertically above a small vessel and focused on a mark on the base of the vessel. A layer of transparent liquid of depth d is poured in the vessel, and then it is found that the image of the mark has been displaced through a distance x which is determined by re-focusing the microscope. Show that the index of refraction of the liquid is equal to $d/(d-x)$.

10. In an actual experiment made by the above method to determine the index of refraction of alcohol, the depth of the liquid was 4 cm., and the displacement of the image was found to be 1.06 cm. What value was found for the index of alcohol? Ans. 1.36.

11. A candle is observed through a tank of water with vertical plane glass walls. The line of sight is perpendicular to the sides of the tank, the candle being 15 cm. from one side and 39 cm. from the opposite side. What is the apparent position of the candle? (Neglect the effect of the thin glass walls.) Ans. It appears to be 9 cm. from the near side.

12. If an object viewed normally through a plate of glass with plane parallel faces seems to be five-sixths of an inch nearer than it really is, how thick is the glass?

Ans. 2.5 inches.

13. A layer of ether 2 cm. deep floats on a layer of water 3 cm. deep. What is the apparent distance of the bottom of

the vessel below the free surface of the ether? (Take index of refraction of water = 1.33 and of ether = 1.36.)

Ans. 3.73 cm.

14. A person looks perpendicularly into a mirror made of plate glass of thickness one-half inch silvered on the back. If his eye is at a distance of 15 inches from the front face, where will his image appear to be?

Ans. $15^2/_3$ inches from the front face.

15. When a stick is partly immersed in a transparent liquid of index n at an angle θ with the free horizontal surface, what is the angle θ' which the part of the stick below the surface appears to make with the horizon as seen by an eye looking vertically down on it from the air above the liquid?

Ans. $\tan \theta' = \dfrac{\tan \theta}{n}$.

CHAPTER V

REFRACTION THROUGH A PRISM

48. Definitions, etc.—An optical prism is a limited portion of a highly transparent substance with polished plane faces where the light is reflected or refracted. Prisms in a great variety of geometrical forms and combinations are employed in many types of modern optical instruments (*cf.* §§ 20, 37); but in this chapter the term *prism* will be restricted to mean a portion of a transparent, isotropic substance included between two polished plane faces that are not parallel. The straight line in which the planes of the two faces meet is called the *edge* of the prism, and the dihedral angle between these planes is called the *refracting angle*. This angle, which will be denoted by the symbol β, may be more precisely defined as *the convex angle through which the first face of the prism has to be turned around the edge of the prism as axis in order to bring this face into coincidence with the second face*. The first face of the prism is that side where the rays enter and the second face is the side from which the rays emerge. Every section made by a plane perpendicular to the edge of the prism is a *principal section*, and we shall consider only such rays as traverse the prism in a principal section, not only because the problem of oblique refraction through a prism presents some difficulties which are beyond the scope of this volume, but especially because in actual practice the principal rays are usually confined to a principal section of the prism. It will also be assumed, for simplicity, that the prism is surrounded by the same medium on both sides.

I. *Geometrical Investigation*

49. Construction of Path of a Ray Through a Prism.—The plane of the diagram (Fig. 73) represents the principal

section of a prism whose edge meets this plane perpendicularly at the point marked V. The traces of the two plane faces are shown by the straight lines Z_1V, Z_2V intersecting at V. The straight line AB_1 represents the path of the given incident ray lying in the plane of the principal section and

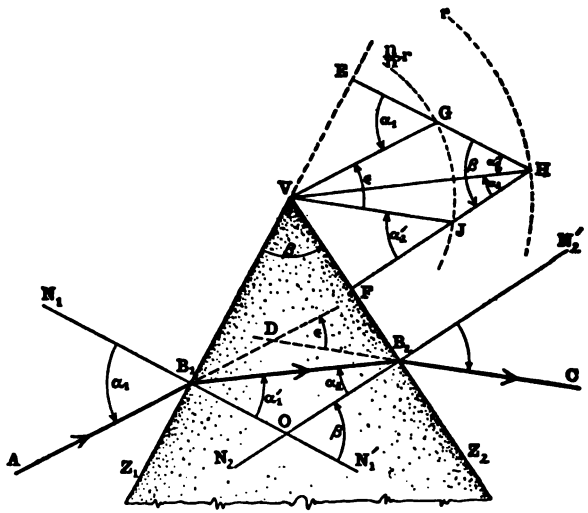


FIG. 73.—Construction of path of ray through principal section of prism ($n' > n$).

falling on the first face of the prism at the incidence-point B_1. The problem of constructing the path of the ray both within the prism and after emergence from it is solved by a method essentially the same as that employed in § 34.

Let n denote the index of refraction of the medium surrounding the prism and n' the index of refraction of the prism-medium itself. With the point V as center, and with radii equal to r and $\frac{n}{n'}.r$, where the radius r may have any convenient length, describe the arcs of two concentric circles both lying within the angle Z_2VE, where E designates a point on the prolongation of the straight line Z_1V beyond V. Through

V draw a straight line VG parallel to AB_1 meeting the arc of radius nr/n' in the point designated by G; and through the point G draw a straight line GE perpendicular at E to the first face of the prism (produced if necessary), and let H designate the point where the straight line GE (likewise produced if necessary) meets the circumference of the other of the two circular arcs. Then the straight line B_1B_2 drawn parallel to the straight line VH will represent the path of the ray within the prism. For if the straight line N_1N_1' is the incidence-normal to the first face of the prism at the point B_1, and if the angles of incidence and refraction at this face are denoted by $\alpha_1 = \angle N_1B_1A$, $\alpha_1' = \angle N_1'B_1B_2$, then by the law of refraction:

$$n . \sin \alpha_1 = n' . \sin \alpha_1'.$$

But by the construction:

$$\frac{\sin \angle EGV}{\sin \angle EHV} = \frac{VH}{VG} = \frac{n'}{n},$$

and since $\angle EGV = \angle N_1B_1A = \alpha_1$, it follows that $\angle EHV = \alpha_1'$; and hence the path of the ray within the prism must be parallel to VH.

Again, from the point H let fall a perpendicular HF on the second face of the prism, where F designates the foot of this perpendicular; and let J designate the point where HF intersects the arc of radius nr/n'. Then the straight line B_2C drawn from the incidence-point B_2 parallel to the straight line VJ will represent the path of the emergent ray. For if we draw N_2N_2' perpendicular to the second face of the prism at B_2, and if the angles of incidence and refraction at this face are denoted by $\alpha_2 = \angle N_2B_2B_1$, $\alpha_2' = \angle N_2'B_2C$, respectively, then $n' . \sin \alpha_2 = n . \sin \alpha_2'$. But

$$\frac{\sin \angle FJV}{\sin \angle FHV} = \frac{VH}{VJ} = \frac{n'}{n},$$

and since by construction $\angle FHV = \alpha_2$, it follows that $\angle FJV = \alpha_2'$, and hence the path of the emergent ray will be parallel to VJ.

The diagram (Fig. 73) is drawn for the case when $n' > n$, as in the ordinary case of a glass prism surrounded by air. The student should draw also a diagram for the other case when $n' < n$, showing the procedure of a ray through a prism of less highly refracting substance than that of the surrounding medium, for example, an air prism surrounded by glass, such as is formed by the air-space between two separated glass prisms.

50. The Deviation of a Ray by a Prism.—The *total deviation* of a ray refracted through a prism, which is equal to the algebraic sum of the deviations produced by the two refractions (§ 35), may be defined as the angle $\epsilon = \epsilon_1 + \epsilon_2$ through which the direction of the emergent ray must be turned in order to bring it into the direction of the incident ray; thus, in Fig. 73, $\epsilon = \angle JVG$; and if the angle ϵ is measured in radians, the arc $JG = \epsilon . JV$. In order to specify completely an angular displacement, it is necessary to give not only the magnitude of the angle and the sense of rotation of the radius vector, but also the plane in which the displacement occurs. This plane may be specified by giving the direction of a line perpendicular to it, which in the case of the angle here under consideration may be the edge of the prism or any line parallel to it; because any such line will be perpendicular to the principal section of the prism in which the ray lies. In fact, the angle ϵ may be completely represented in a diagram by a straight line drawn parallel to the edge of the prism, which by its length indicates the magnitude of the angle and by its direction shows the sense of rotation. Thus, for example, the line may be drawn along the edge of the prism itself from a point V in the plane of the principal section and always in such a direction that on looking along the line towards that plane $\angle JVG = \epsilon$ will be seen to be a counter-clockwise rotation. A deviation of 20° in a principal section coinciding, say, with the plane of the paper would be represented, therefore, by a straight line perpendicular to this plane of length 20 cm., if each degree

were to be represented by one centimeter. If $\epsilon = +20°$, this line would point out from the paper towards the reader, and if $\epsilon = -20°$, it would point away from him. Thus, if the prism, originally "base down," is turned "base up" (as the opticians say), everything else remaining the same, the sign of the angle ϵ will be reversed, and so also will be the direction of the vector which represents this angle.

51. Grazing Incidence and Grazing Emergence.—The angle GHJ between the normals to the two faces of the prism is equal to the refracting angle β; and hence for a given prism this angle will remain always constant. No matter how the direction of the incident ray AB_1 (or VG) may be varied, the vertex H of this angle will lie always on a certain portion of the circumference of the construction-circle of radius r, and the sides HG, HJ will remain always in the same fixed directions perpendicular to the faces of the prism. Obviously, there will be two extreme or limiting positions of the point H marking the ends of the arc on which it is confined, namely, the positions which H has when one of the sides of the angle GHJ is tangent to the circle of radius nr/n'; which can occur only for the case when $n' > n$, because otherwise the point H will lie inside the circumference of this circle and therefore it will be impossible for either HG or HJ to be tangent to it.

If the side HG is tangent to the inner circle at G, as shown in Fig. 74, the point G will lie in the plane of the first face of the prism, and accordingly the corresponding ray incident on the first face of the prism at the point B_1, which must have the direction VG, will be the ray Z_1B_1 which, entering the prism at "*grazing*" *incidence* ($a_1 = 90°$), traverses the prism as shown in the figure.

On the other hand, when the side HJ of the angle GHJ is tangent at J to the construction-circle of radius nr/n' (Fig. 75), the point J will lie in the second face of the prism, and the straight line VJ will coincide with the straight line VZ_2. Under these circumstances the ray emerges from the prism at B_2 along the second face in the direction B_2Z_2 ($a_2' = -90°$).

The straight line KB_1 shows the path of the ray incident on the first face of the prism at B_1 which "grazes" the second face on emerging from the prism. Any ray incident at B_1 and lying in the principal section of the prism within the angle KB_1Z_1 will succeed in getting through the prism and emerging.

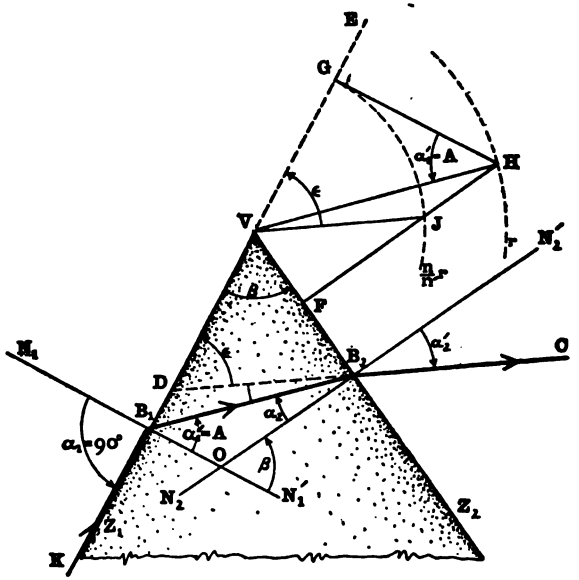


FIG. 74.—Case when ray "grazes" first face of prism.

into the surrounding medium again; whereas if the ray incident at B_1 lies anywhere within the angle VB_1K, it will be *totally reflected* at the second face of the prism. The ray KB_1 is called the *limiting incident ray* and $\angle N_1B_1K = \iota$ is the *limiting angle of incidence*. These relations will be discussed more fully in the analytical investigation of the path of a ray through a prism (§§ 55, foll.); but it may be remarked that $\angle GHV = \alpha_1'$ in Fig. 74 and $\angle JHV = \alpha_2$ in Fig. 75 are both equal to the critical angle A (§ 36) with respect to the two media n, n' ($\sin A = n/n'$).

52. Minimum Deviation.—Between the two extreme or terminal positions of the vertex H of ∠GHJ shown in Figs. 74 and 75, there is also an intermediate place which is of special interest and importance and to which, therefore, attention must be called. In general, the sides HG, HJ inter-

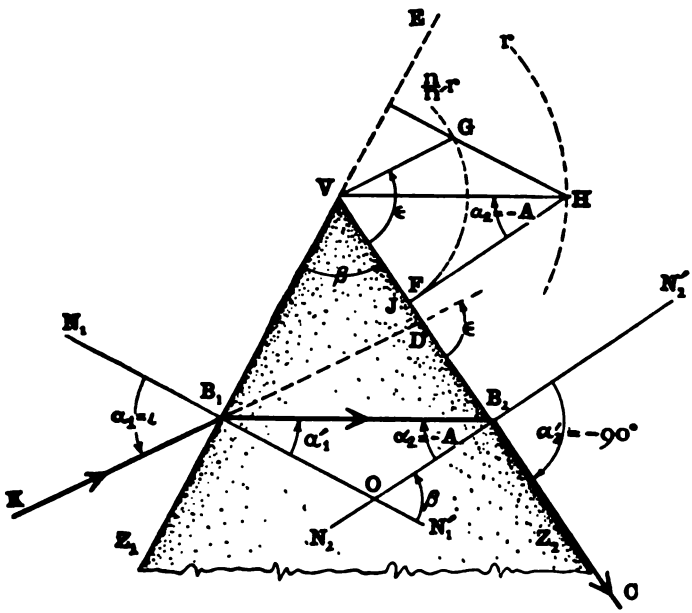


FIG. 75.—Case when ray "grazes" second face of prism.

cepted between the two construction-circles (Fig. 73) will be unequal in length, but if HG = HJ, as in Fig. 76, the angles GVJ, GHJ and EVF will evidently all be bisected by the diagonal VH of the quadrangle VGHJ. When this happens, the path B_1B_2 of the ray inside the prism, which is parallel to VH, *crosses the prism symmetrically*, that is, the triangle VB_1B_2 is isosceles. In fact, the points designated in the diagram by the letters V, D and O will be the summits of isosceles triangles having the common base B_1B_2, and they will all lie therefore on the bisector of the refracting angle $\beta =$

$\angle Z_1VZ_2$, which is perpendicular to VH. The angle of incidence at the first face and the angle of emergence at the second face are equal in magnitude, although they are described in opposite senses, so that $a_2' = -a_1$. The same is true also in regard to the angles which the ray makes inside

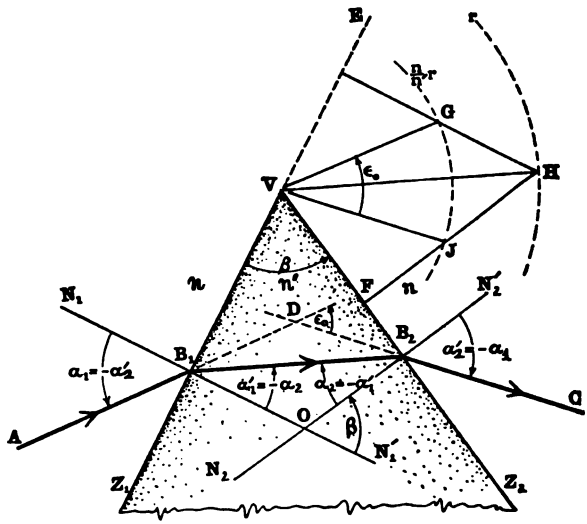


FIG. 76.—Ray traverses prism symmetrically ($VB_1 = VB_2$); case of minimum deviation.

the prism with the normals to the two faces, that is, $a_2 = -a_1'$.

Now when the ray traverses the prism symmetrically, as represented in Fig. 76, the deviation ϵ has its least value ϵ_0. In order to show that this is true, it will be convenient to reproduce the symmetrical quadrangle VGHJ in Fig. 76 in a separate diagram, as in Fig. 77. Suppose that H′ designates the position of a point infinitely near to H lying likewise on the arc of the circle of radius r, and draw H′G′, H′J′ parallel to HG, HJ and meeting the arc of the other circle in the points G′, J′, respectively. In the figure the point H′ is taken *below* the point H, and in this case it is plain that

the two parallels HJ, H′J′ will meet the circumference of the inner circle more obliquely than the other pair of parallel lines HG, H′G′, and, consequently, the infinitely small arc J′J intercepted between the first pair will be greater than the arc G′G intercepted between the second pair. Hence, the small angle J′VJ will be greater than ∠G′VG, and therefore

$$\angle J'VG' > \angle JVG.$$

The angle JVG here is the angle of deviation (ϵ_0) of the ray that goes symmetrically through the prism; whereas $\angle J'VG' = \epsilon$ is the angle of deviation of a ray which traverses the prism along a very slightly different path. And according to the above reasoning (for we shall arrive at the same result if we take the point H′ also above H), we find:

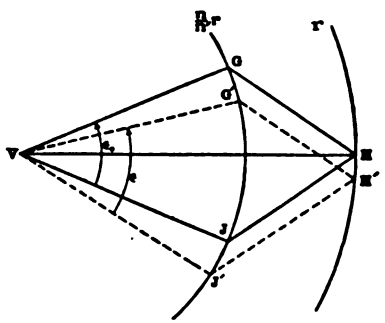


FIG. 77.—Case of minimum deviation.

$$\epsilon > \epsilon_0.$$

Accordingly, we see that *the ray which traverses the prism symmetrically in the plane of a principal section is also the ray which is least deviated.*

It is easy to verify this statement experimentally. Thus, for example, if a bundle of parallel rays is allowed to fall on an isosceles triangular prism, so that while some of the rays are incident on one of the equal faces and are transmitted through the prism, the other rays of the bundle are reflected from the base of the prism, as represented in (1) in Fig. 78; and if then the prism is gradually turned around an axis parallel to its edge, first, into position (2), which is the position of minimum deviation, and then past this position into a third position (3), it will be observed that when the prism is in the position of minimum deviation the rays reflected from the base will be parallel to the rays which emerge at

the second face of the prism; which can only be the case when the rays cross the prism symmetrically.

In spectroscopic work and in many other scientific uses of the prism, the position of minimum deviation, which is easily found, is frequently the most convenient and advantageous adjustment of the prism for purposes of observation.

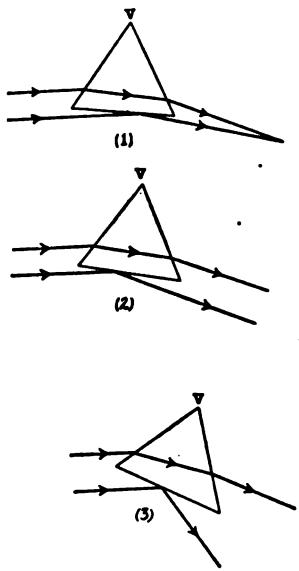


FIG. 78.—Experimental proof that ray which traverses prism symmetrically is ray of minimum deviation.

53. Deviation away from the Edge of the Prism.—*When a ray of light passes through a prism of more highly refracting material than that of the surrounding medium* ($n' > n$), *the deviation is always away from the edge towards the thicker part of the prism.*

If the angles of the triangle VB_1B_2 (Fig. 79) at B_1 and B_2 are both acute, the incident and emergent rays lie on the sides of the normals at B_1 and B_2 away from the prism-edge, so that at both refractions the ray will be bent away from the edge. If one of the angles, say, the angle at B_2, is a right angle, the ray will not be deviated at all by the refraction at this point, but at the other incidence-point it will be bent away from the edge. And, finally, if one of the angles at B_1 or B_2 is obtuse, for example, the angle at B_1 (Fig. 80), the deviation on entering the prism will, it is true, be towards the edge of the prism, but this deviation will not be so great as the subsequent deviation away from the edge which is produced at the second refraction when the ray issues from the prism, as may be easily seen from the diagram. Thus,

in every case when $n'>n$, the total deviation will be away from the prism-edge.

If $n'<n$, all these effects will be reversed.

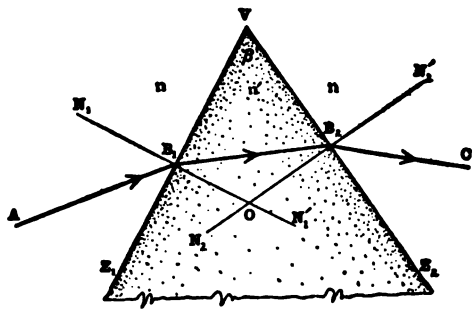


FIG. 79.—Deviation away from edge of prism.

54. Refraction of a Plane Wave Through a Prism.—The diagram (Fig. 81) shows a principal section of the prism, and the straight line B_1D represents the trace of a plane wave (supposed to be perpendicular to the plane of the paper and parallel therefore to the edge of the prism) advancing towards the first face of the prism in the direction DV at right angles to B_1D. If around the point B_1, which lies in the first face of the prism, the arc of a circle is described with radius $BE=\frac{n}{n'}DV$, then, according to HUYGENS'S principle (§ 5), the straight line VE tangent to this circle at E will represent the trace of the wave-front inside the prism. Let the straight line B_1E meet the second face of the prism at B_2. Around V as center with radius $VF=\frac{n'}{n}EB_2$ describe the arc of a circle; then the straight line B_2F tangent to this

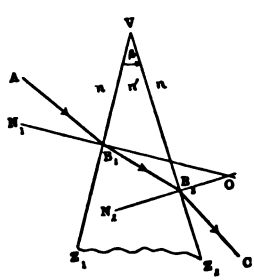


FIG. 80.—Deviation away from edge of prism.

circle at F will represent the trace of the emergent wave-front.

The disturbance at any point C will have emanated from some point on AB_1, and the time taken by the light to go from B_1 to B_2 inside the prism will be the same as that required to go from D to F in the surrounding medium (§ 39); that is, the optical lengths along these two routes are equal. For, as appears from the construction,

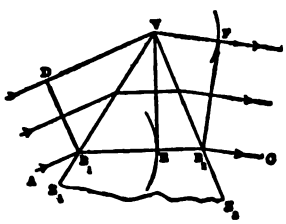


FIG. 81.—Refraction of plane wave through prism.

$$n(DV+VF)=n'.B_1B_2.$$

An excellent and most instructive mechanical illustration of the refraction of a plane wave through a prism can be obtained by using the roller and tilted board described in § 32 with a triangular piece of plush cloth glued in the middle of the board to represent the prism (see Fig. 45).

II. *Analytical Investigation*

55. Trigonometric Calculation of the Path of a Ray in a Principal Section of a Prism.—The angles of incidence and refraction at the first and second faces of the prism, denoted by a_1, a_1' and a_2, a_2', are, by definition (§ 27), the acute angles through which the normals to the refracting surfaces at the incidence-points have to be turned in order to bring them into coincidence with the incident and refracted rays at the two faces of the prism; thus, in Fig. 73, $\angle N_1B_1A = a_1$, $\angle N_1'B_1B_2 = a_1'$, $\angle N_2B_2B_1 = a_2$, $\angle N_2'B_2C = a_2'$.

Assuming that the prism is surrounded by the same medium on both sides, and being careful to note the sense of rotation of each of the angles, we obtain by the law of refraction, taken in conjunction with the obvious geometrical relations as shown in the figure, the following system of

equations for calculating the path of a ray through a principal section of a prism:

$$n'.\sin\alpha_1' = n.\sin\alpha_1, \quad \alpha_2 = \alpha_1' - \beta, \quad n.\sin\alpha_2' = n'.\sin\alpha_2.$$

Combining these formulæ so as to eliminate α_1' and α_2, we may derive the following convenient expression for determining the angle of emergence (α_2') at the second face of the prism:

$$\sin\alpha_2' = \sin\alpha_1.\cos\beta - \sin\beta\frac{\sqrt{n'^2 - n^2.\sin^2\alpha_1}}{n}.$$

Thus, if we know the value of the relative index of refraction (n'/n) and the refracting angle of the prism ($\beta = \angle Z_1VZ_2$), we can calculate the angle of emergence (α_2') corresponding to any given direction (α_1) of the ray incident on the first face of the prism.

The total deviation (ϵ) of a ray refracted through a prism is measured, as defined above (§ 50), by $\angle JVG$, and since this angle is equal to the external angle at D in the triangle DB_1B_2, we have:

$$\begin{aligned}\epsilon &= \angle B_2B_1D + \angle DB_2B_1 \\ &= \angle N_1'B_1D - \angle N_1B_1B_2 + \angle DB_2N_2 - \angle B_1B_2N_2 \\ &= \alpha_1 - \alpha_1' - \alpha_2' + \alpha_2;\end{aligned}$$

and since $\alpha_1' - \alpha_2 = \beta$, we obtain finally the following expression for the angle of deviation:

$$\epsilon = \alpha_1 - \alpha_2' - \beta.$$

These formulæ contain the whole theory of the refraction of a ray through a prism in a principal section. It will be interesting to discuss analytically some of the special cases which we have already studied in the preceding sections of this chapter.

56. Total Reflection at the Second Face of the Prism.— If the angle of emergence at the second face of the prism is a right angle, that is, if $\alpha_2' = -90°$, the emergent ray B_2C will issue from the prism along the second face in the direction B_2Z_1 (Fig. 75). Hence, $\sin\alpha_2 = \frac{n}{n'}.\sin\alpha_2' = -\frac{n}{n'}$, and therefore $\alpha_2 = -A$, where A denotes the critical angle (§ 36) of the

media n, n', defined by the relation $\sin A = \frac{n}{n'}$. If the absolute value of the angle a_2 is greater than A, the ray will be *totally reflected* at the second face of the prism, and there will be no emergent ray. This case may be discussed in some detail.

For a prism of given refracting angle (β), there is a certain limiting value (ι) of the angle of incidence (a_1) at the first face of the prism (§ 51) for which we shall have at the second

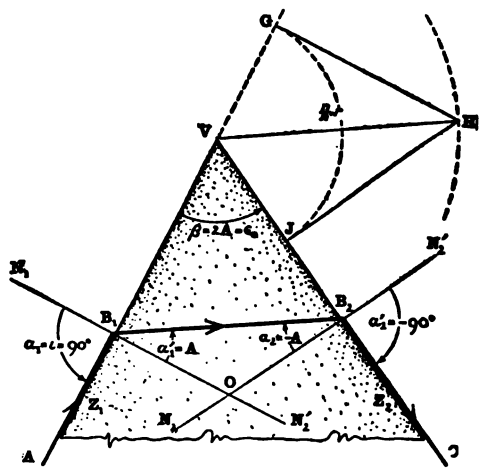


FIG. 82.—Prism with refracting angle $\beta = 2A$.

face the values $a_2 = -A$, $a_2' = -90°$; so that *a ray which is incident on the first face of the prism at an angle less than the limiting angle ι will not pass through the prism but will be totally reflected at the second face.* Putting $a_2 = -A$, we find $a_1' = \beta - A$, and therefore, since $a_1 = \iota$,

$$\sin \iota = \frac{n'}{n} \sin (\beta - A),$$

which is the trigonometric formula for computing the value of the limiting angle of incidence for a given prism. It will

be worth while to examine this formula for certain particular values of the refracting angle β.

(1) If $\beta > 2A$, then, since $\sin A = \frac{n}{n'}$, the formula shows that $\sin \iota$ will be greater than unity, so that for a prism of this form there is no angle corresponding to the limiting angle ι. *No ray can be transmitted through a prism whose refracting angle is more than twice as great as the critical angle of the two media in question.* A prism of this size is called a *totally*

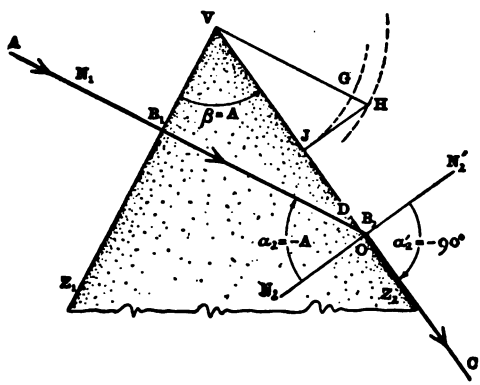


FIG. 83.—Prism with refracting angle $\beta = A$.

reflecting prism; if it is made of glass of index 1.5 and surrounded by air, the refracting angle should be about 84° at least.

(2) If $\beta = 2A$, we find that $\iota = 90°$; which is the case represented in Fig. 82. The only ray that can get through this prism is the ray that traverses it symmetrically, entering the prism along one face and leaving it along the other.

(3) If $\beta > A$ but $< 2A$ (that is, if $2A > \beta > A$), the value of the angle ι as determined by the formula above will be comprised between 90° and 0°. This is the case which was shown in Fig. 73. The direction of the limiting incident ray is be-

tween Z_1B_1 and N_1B_1; that is, $\angle V_1BK$ will be an obtuse angle.

(4) If $\beta = A$, we find $\iota = 0°$, and then the limiting incident ray will proceed along the normal N_1B_1, as shown in Fig. 83, and $\angle VB_1K$ (or $\angle VB_1A$) will be a right angle.

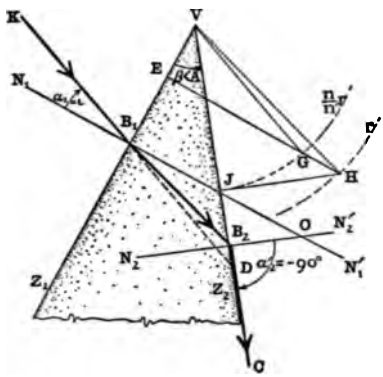


FIG. 84.—Prism with a refracting angle $\beta < A$.

(5) Finally, if $\beta < A$, the limiting angle of incidence (ι) will be negative in sign; and therefore in a more or less thin prism of this description the limiting incident ray KB_1 will fall on the side of the normal N_1B_1 towards the apex V of the prism, so that the angle VB_1K will be an acute angle (Fig. 84).

Any ray incident on the first face of the prism at B_1 and lying within the angle KB_1Z_1 will be transmitted through the prism; whereas if the ray falls within the supplementary angle VB_1K, it will be totally reflected at the second face.

In KOHLRAUSCH'S method of measuring the relative index of refraction ($\frac{n'}{n}$), the prism is adjusted so that the incident ray "grazes" the first face, and then if the refracting angle of the prism (β) is known, and if the angle of emergence (a_2') is measured, the value of $n' : n$ may be calculated by means of the formula:

$$\sqrt{\left(\frac{n'}{n}\right)^2 - 1} = \frac{\cos\beta - \sin a_2'}{\sin\beta}, \quad (a_1 = 90°).$$

The principle of total reflection is also employed in the prism refractometers of ABBE and PULFRICH for measurement of the index of refraction.

57. Perpendicular Emergence at the Second Face of the Prism.—For this case we have $a_2 = a_2' = 0°$, and therefore $a_1' = \beta$, $a_1 = \beta - \epsilon$, and hence:

$$\frac{n'}{n} = \frac{\sin(\beta - \epsilon)}{\sin \beta};$$

which is also a convenient formula for the experimental determination of the value of the relative index of refraction. A description of the apparatus and the method of procedure may be found in the standard treatises on physics.

58. Case when the Ray Traverses the Prism Symmetrically.—As has been pointed out already (§ 52), a special case of great interest occurs when the ray traverses the prism symmetrically. Under these circumstances, the general prism-equations given in § 55 take the following forms:

$$a_1 = -a_2' = \frac{\beta + \epsilon_0}{2}, \quad a_1' = -a_2 = \frac{\beta}{2},$$

$$\frac{n'}{n} = \frac{\sin \frac{\beta + \epsilon_0}{2}}{\sin \frac{\beta}{2}};$$

where ϵ_0 denotes the angle of deviation of this symmetric ray. The last of these formulæ is the basis of the FRAUNHOFER method of determining the relative index of refraction, the angles β and ϵ_0 being both capable of easy measurement.

This last formula may also be transformed into the following form:

$$\tan \frac{\beta}{2} = \frac{n . \sin \frac{\epsilon_0}{2}}{n' - n . \cos \frac{\epsilon_0}{2}};$$

whereby the refracting angle β can be calculated in terms of n, n' and ϵ_0.

59. Minimum Deviation.—The prism itself is defined by its refracting angle (β) and the relative index of refraction (n'/n). The total deviation (ϵ) of a ray refracted through

a given prism depends only on the angle of incidence (a_1), according to the formula:

$$\epsilon = a_1 - a_2' - \beta;$$

for the angle a_2' may be expressed in terms of a_1, β and n'/n, as we have seen (§ 55). Hence, for a given value of these three magnitudes the angle ϵ will be uniquely determined. On the other hand, for a given value of the angle ϵ there will always be *two* corresponding values of the angle of incidence a_1; for it is obvious from the principle of the reversibility of the light-path (§ 29) that a second ray incident on the first face of the prism at an angle equal to the angle of emergence of the first ray will emerge at the second face at an angle equal to the angle of incidence of the first ray at the first face, and these two rays will be equally deviated in passing through the prism. For example, suppose that the values of the angles of incidence and emergence in the case of the first ray are $a_1 = \gamma$, $a_2' = \gamma'$: a second ray incident on the first face of the prism at the angle $a_1 = -\gamma'$ will emerge at the second face at an angle $a_2' = -\gamma$, and each of these rays will suffer precisely the same deviation, viz., $\epsilon = \gamma - \gamma' - \beta$. Thus, corresponding to any given value of the angle ϵ, within certain limits, there will always be a pair of rays which are deviated by this same amount. One pair of such rays consists of the *two identical rays* determined by the relation

$$a_1 = \gamma = -a_2'.$$

In fact, this is the ray which traverses the prism symmetrically, and a little reflection will show that the deviation of this ray must be either a maximum or a minimum.

But while the best way of demonstrating that *the ray which goes symmetrically through the prism is the ray of minimum deviation* (§ 52) involves the employment of the methods of the differential calculus, the following analytical proof demands of the student a knowledge of only elementary mathematics.

The deviation at the first face of the prism is $\epsilon_1 = a_1 - a_1'$,

and that at the second face is $\epsilon_2 = a_2 - a_2'$ (§ 35), and hence the total deviation is

$$\epsilon = \epsilon_1 + \epsilon_2 = (a_1 - a_1') + (a_2 - a_2'),$$

or, since $a_1' - a_2 = \beta$, $\epsilon = a_1 - a_2' - \beta$, as has been already remarked, for example, in § 55. *Assume now that* $n' > n$, and, consequently, that *the angle* ϵ *is positive*, as is always the case when the ray is bent away from the edge of the prism (§ 53); then it is evident that the angle ϵ will have its least value (ϵ_0) in the case of that ray for which the function $(a_1 - a_2')$ is least. Now since

$$n . \sin a_1 = n' . \sin a_1', \quad n . \sin a_2' = n' . \sin a_2,$$

we obtain by subtraction:

$$n(\sin a_1 - \sin a_2') = n'(\sin a_1' - \sin a_2),$$

and hence by an obvious trigonometric transformation:

$$n . \sin\frac{a_1 - a_2'}{2} . \cos\frac{a_1 + a_2'}{2} = n' . \sin\frac{a_1' - a_2}{2} . \cos\frac{a_1' + a_2}{2},$$

which may be written as follows:

$$\sin\frac{a_1 - a_2'}{2} = \frac{n'}{n} . \sin\frac{\beta}{2} \cdot \frac{\cos\dfrac{a_1' + a_2}{2}}{\cos\dfrac{a_1 + a_2'}{2}}.$$

According as $a_1 \gtreqless - a_2'$, the deviation ϵ_1 at the first face of the prism will (see § 35) be greater than, equal to, or less than, the deviation ϵ_2 at the second face; that is, according as $a_1 \gtreqless - a_2'$, we shall have $(a_1 - a_1') \gtreqless (a_2 - a_2')$, and hence also

$$\frac{a_1 + a_2'}{2} \gtreqless \frac{a_2 + a_1'}{2}.$$

If we suppose, first, that $a_1 > - a_2'$, then $a_1' > - a_2$ and $(a_2 + a_1') > 0$; and since the cosine of a positive angle decreases as the angle increases, it follows that here we must have:

$$\cos\frac{a_1' + a_2}{2} > \cos\frac{a_1 + a_2'}{2}.$$

On the other hand, if we suppose, second, that $a_1 < - a_2'$, then

$a_1' < -a_2$ and $(a_2 + a_1') < 0$; but in this case $(a_2 + a_1') > (a_1 + a_2')$, so that although $(a_2 + a_1')$ and $(a_1 + a_2')$ are both negative, the absolute value of the former is greater than that of the latter, and hence here also we find exactly the same result as before.

Thus, whether a_1 is greater or less than $-a_2'$, the ratio

$$\frac{\cos\frac{a_1' + a_2}{2}}{\cos\frac{a_1 + a_2'}{2}} > 1;$$

and only in the case when $a_1 = -a_2'$ will this ratio equal to unity. Hence, $\sin\frac{a_1' - a_2'}{2}$ has its least value when $a_1 = -a_2'$, and then also the deviation (ϵ) is a minimum and equal to

$$\epsilon_0 = 2a_1 - \beta.$$

The same process of reasoning applied to the case when $n' < n$ leads to the conclusion that the angle ϵ will be a maximum for the ray which traverses such a prism symmetrically, for example, an air-prism surrounded by glass; but in this case the angle ϵ will be negative in sign, and since a maximum value of a negative magnitude corresponds to a minimum absolute value, the actual deviation of the ray is least in this case also.

60. Deviation of Ray by Thin Prism.—If the refracting angle of the prism (β) is small, as represented, for example, in Fig. 85, the deviation (ϵ) will likewise be a small angle of the same order of smallness; for if $\beta = a_1' - a_2$ is small, then $(a_1 - a_2')$ will be small also, and the angle ϵ is the difference between these two small magnitudes. In fact, the deviation ϵ produced by a thin prism will not only always be small, but it will never be very different from its minimum value ϵ_0. Accordingly, in the case of a thin prism, we may put $\epsilon = \epsilon_0$ without much error; and therefore very approximately (see § 58):

$$\frac{n'}{n} = \frac{\sin\frac{\epsilon + \beta}{2}}{\sin\frac{\beta}{2}}.$$

Consequently, the deviation ϵ, as calculated by this formula, will depend only on the prism-constants (β, n': n) and not on the angle of incidence (α_1). The smaller the angle β, the more nearly correct this formula will be; and if the angle β is so small that we may substitute $\frac{\beta}{2}$ and $\frac{\epsilon+\beta}{2}$ in place of $\sin\frac{\beta}{2}$ and $\sin\frac{\epsilon+\beta}{2}$, respectively, we obtain the exceedingly

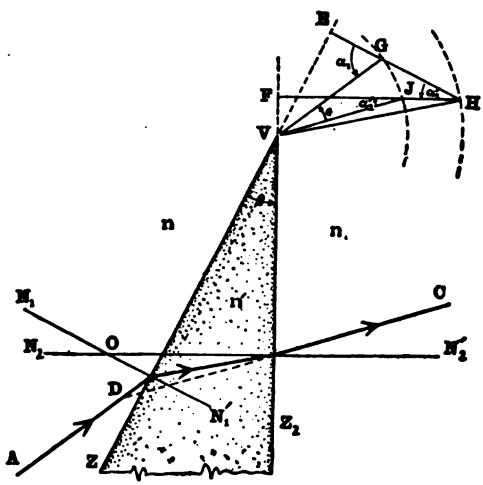


FIG. 85.—Prism with comparatively small refracting angle.

useful and convenient practical relation for the angle of deviation of a ray refracted through a thin prism, viz.:

$$\epsilon = (\frac{n'}{n} - 1)\beta,$$

which, however, is more frequently written:

$$\epsilon = (n-1)\beta,$$

where n is employed now to denote the relative index of refraction. Accordingly, *in a thin prism the deviation is di-*

rectly proportional to the refracting angle. For example, the deviation in the case of a thin glass prism surrounded by air for which $n = 1.5$ is one-half the refracting angle.

61. Power of an Ophthalmic Prism. Centrad and Prism-Dioptry.—An ophthalmic prism is a thin glass prism, whose index of refraction is usually about 1.52, which is used to correct faulty tendencies and weaknesses of the ocular muscles which turn the eye in its socket about the center of rotation of the eye-ball. In an ordinary laboratory prism the two faces are usually cut in the form of rectangles having the edge of the prism as a common side; but the contour of an ophthalmic prism which has to be worn in front of the eye in a spectacle-frame is circular or elliptical like that of any other eye-glass, and its edge is the line drawn tangent to this curve at the thinnest part of the glass. The line drawn perpendicular to this tangent at the point of contact and lying in the plane of one of the faces of the prism is the so-called "*base-apex*" *line*, which is a term frequently employed by writers on spectacle-optics.

The formula

$$\epsilon = (n-1)\beta$$

obtained in § 69 is peculiarly applicable to the weak prisms used in spectacles. As long as the refracting angle of the prism does not exceed, say, 10°, the error in the value of ϵ as calculated by this approximate formula will be less than 5 per cent.

Formerly it was customary to give the strength or power of an ophthalmic prism in terms of its refracting angle β expressed in degrees; but the proper measure of this power is the deviation produced by the prism. However, instead of measuring this angle in degrees, DENNETT has suggested that the deviation of an ophthalmic prism shall be measured in terms of a unit angle called a *centrad*, which is the one-hundredth part of a radian and equal therefore to the angle subtended at the center of a circle of radius one meter by an arc of length one centimeter. Since π radians = 180°, the

relation between the centrad and the degree is given as follows:

$$1° = \frac{100\,\pi}{180}\text{centrads},$$

or

$$1° = 1.745 \text{ ctrd.}, \quad 1 \text{ ctrd.} = 0.573°.$$

Prior to this suggestion, Mr. C. F. PRENTICE, of New York, had proposed in 1888 to measure the deviation of an ophthalmic prism in terms of the linear or tangential displacement in centimeters on a screen placed at a distance of one meter from the prism. If the straight lines OA, OB (Fig. 86) represent the directions of the incident and emergent rays, respectively, then $\angle AOB$ will be the angle of deviation of the prism; and if a plane screen placed at right angles to OA at A is intersected by OB at B, then $\tan\angle AOB = \frac{AB}{OA}$. Now if the distance OA = 100 cm. and if AB = x cm., then, according to PRENTICE's method, the $\angle AOB$ would be an angle of x units and the power of the prism would be denoted by x. Dr. S. M. BURNETT suggested that the name *prism-diopter* or *prism-dioptry* be given to this unit. (The term "prismoptrie" was proposed by Professor S. P. THOMPSON.) The prism-dioptry is the angle corresponding to a deviation of one centimeter on a tangent line at a distance of one meter; and, accordingly, when the angle of deviation is equal to the angle whose trigonometric tangent is $x/100$, the power of the prism is said to be x prism-dioptries or $x\triangle$, where the symbol $\triangle$ stands for prism-dioptry. The chief objection to be urged against this unit of angular measurement is that the angle subtended at a given point O (Fig. 87) by equal line-segments on a line Ay perpendicular to Ox at A diminishes as the

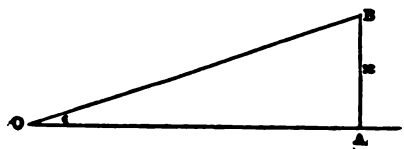


FIG. 86.—Deviation of prism: tan ϵ = AB : OA.

segment on Ay is taken farther and farther from A. In other words, since $\tan^{-1} x/100$ is less than $x.\tan^{-1} 1/100$, x prism-dioptries is less than x times one prism-dioptry. Ordinarily, the variability in the magnitude of a unit would constitute an insuperable objection to it; but so long as the

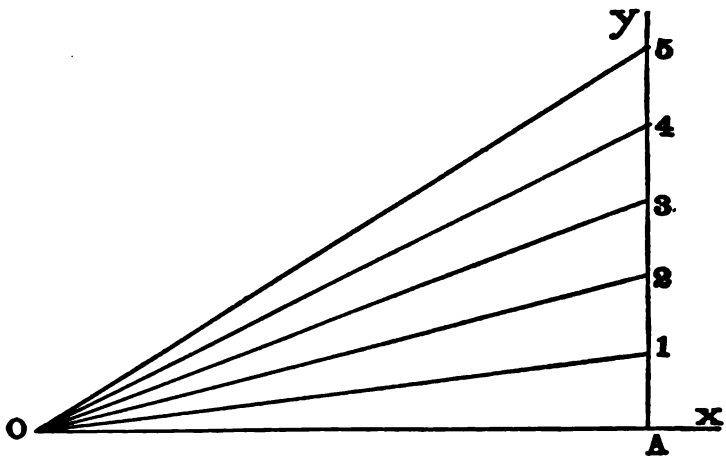


FIG. 87.—Unequal angles subtended at O by equal intervals on straight line Ay drawn perpendicular to OA.

angles to be measured are always small, as is the case with ophthalmic prisms, the prism-dioptry may be regarded as invariably equal to the $\tan^{-1} 1/100$ or about 34′ 22.6″ without sensible error; and hence we may say, for example, that $2\triangle + 3\triangle = 5\triangle$, although this statement is not quite accurate. At any rate, whatever may be the theoretical objections, this unit of measurement of the strength of a thin prism is so convenient and satisfactory that it has been generally adopted in ophthalmic practice.

In point of fact, with the small angular magnitudes which are here pre-supposed (the power of an ophthalmic prism seldom exceeds 6 ctrd.), there is practically no distinction to be made between the angle itself and the tangent of the angle,

so that *we may regard the centrad and the prism-dioptry as identical in most cases;* that is,

$$1\triangle = 1 \text{ ctrd.} = 0.573°.$$

Accordingly, we obtain the following relation between the power (p) of an ophthalmic prism expressed in prism-dioptries or centrads and the refracting angle (β) given in degrees:

$$p = \frac{100\,\pi}{180}(n-1)\beta = 1.745(n-1)\beta,$$

where n denotes the relative index of refraction. If $n = 1.5$, then the power of a prism of refracting angle β degrees is 0.873 prism-dioptries.

However, in order to exhibit the actual relations still more clearly, the following table gives the values in degrees, minutes and seconds of all integral numbers of prism-dioptries and centrads from 1 to 20; and incidentally it will be seen that whereas an angle of k centrads contains k times as many degrees, minutes and seconds as an angle of 1 centrad, where k denotes any integer from 1 to 20, the same statement is not strictly true of the prism-dioptry.

Prism-Dioptries	*Equivalent in degrees, minutes and seconds*	*Centrads*	*Equivalent in degrees, minutes and seconds*
1	0° 34′ 22.6″	1	0° 34′ 22.7″
2	1° 8′ 44.8″	2	1° 8′ 45.3″
3	1° 43′ 6.1″	3	1° 43′ 8.0″
4	2° 17′ 26.2″	4	2° 17′ 30.6″
5	2° 51′ 44.7″	5	2° 51′ 53.3″
6	3° 26′ 1.1″	6	3° 26′ 15.9″
7	4° 0′ 15.0″	7	4° 0′ 38.6″
8	4° 34′ 26.1″	8	4° 35′ 1.2″
9	5° 8′ 33.9″	9	5° 9′ 23.9″
10	5° 42′ 38.1″	10	5° 43′ 46.5″
11	6° 16′ 38.3″	11	6° 18′ 9.2″
12	6° 50′ 34.0″	12	6° 52′ 31.8″
13	7° 24′ 24.9″	13	7° 26′ 54.5″
14	7° 58′ 10.6″	14	8° 1′ 17.1″
15	8° 31′ 50.8″	15	8° 35′ 39.8″
16	9° 5′ 25.0″	16	9° 10′ 2.4″
17	9° 38′ 53.0″	17	9° 44′ 25.1″
18	10° 12′ 14.3″	18	10° 18′ 47.7″
19	10° 45′ 28.7″	19	10° 53′ 10.4″
20	11° 18′ 35.8″	20	11° 27′ 33.0″

62. Position and Power of a Resultant Prism Equivalent to Two Thin Prisms.—In ascertaining the prismatic correction of the eye of a patient, the oculist or optometrist sometimes finds it convenient and advantageous to employ a combination of two thin prisms placed one in front of the

other with their edges inclined to each other at an angle γ which can be measured; and having obtained the necessary correction in this way, he has to prescribe a single prism which will produce precisely the same resultant effect as the two superposed prisms of the trial-case. In general, it would be exceedingly laborious and difficult to calculate the power of this resultant prism, but, fortunately, the problem in this case is enormously simplified by the fact that the refracting angles are so small that it is quite simple to obtain an approximate solution which is sufficiently accurate and reliable for ordinary practical purposes.

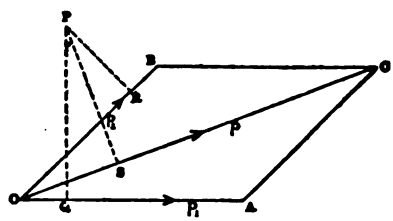


FIG. 88, *a*.—Parallelogram law for finding single prism equivalent to a combination of two thin prisms.

Let the deviation-angles or powers of the two prisms, denoted by p_1 and p_2, be represented, according to the method explained in § 50, by the vectors OA, OB, respectively (Fig. 88), which are drawn parallel to the edges of the prism, so that $\angle AOB = \gamma$. Complete the parallelogram OACB and draw the diagonal OC. The vector OC will represent on the same scale the deviation-angle or power p of the resultant prism, as we shall proceed to show.

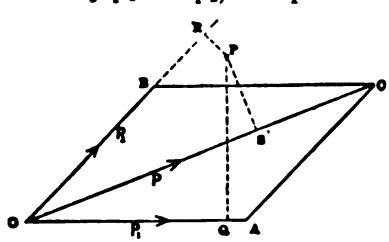


FIG. 88, *b*.—Parallelogram law for finding single prism equivalent to a combination of two thin prisms.

If a point P is taken anywhere in the plane of the parallelogram OACB, it may easily be proved that the area of the triangle POC is equal to the sum or difference of the areas of the triangles POA and POB according as the point P lies

outside the $\angle$AOB, as in Fig. 88 (*a*), or inside this angle, as in Fig. 88 (*b*), respectively. And, therefore, if PQ, PR and PS are drawn perpendicular to OA, OB and OC, respectively, then

$$SP.OC = QP.OA \pm RP.OB.$$

For simplicity, let us assume that the deviations p_1, p_2 produced by the two component prisms are indefinitely small. Now suppose that the point P is turned, first, about OA as axis through a very small angle p_1 and then about OB as axis through the small angle p_2. In consequence of the first rotation it will move perpendicularly out from the plane of the paper towards the reader through a tiny distance corresponding to the arc of a circle described around Q as center with radius QP, the length of this arc being equal to the product of the radius by the angle, that is, equal to QP.OA, since the length of OA is made equal to the magnitude of the angle p_1. If now in this slightly altered position the point P is again rotated, this time, however, around OB as axis, through another small angular displacement $p_2 = OB$, either it will move a little farther out from the plane AOB, as in the case shown in Fig. 88 (*a*), or it will move back away from the reader, as in the case shown in Fig. 88 (*b*), by an additional amount equal to RP.OB. And as this latter displacement will also be very nearly at right angles to the plane of the paper, the resultant angular displacement of the point P may be regarded as equal to the algebraic sum of its two successive displacements and numerically equal, therefore, to

$$QP.OA \pm RP.OB,$$

where the upper sign is to be taken in case the point P lies outside the angle AOB and the lower sign in case it lies inside this angle. In either case, therefore, the resultant displacement of P will be equal to SP.OC. But this product is equal to the linear displacement which the point P would have if it experienced an angular displacement represented by the vector OC.

Hence, if the straight lines OA, OB drawn parallel to the

edges of the two thin prisms represent the components of the total deviation of a ray which traverses both prisms, the diagonal OC of the parallelogram OABC will represent the resultant or total deviation, and this effect will be produced by a single prism of power $p = \text{OC}$ placed with its edge inclined to the edge of the prism of power p_1 ($= \text{OA}$) at an angle $\theta = \angle \text{OAC}$. If the powers p_1, p_2 of the two component prisms are given in prism-dioptries (or in terms of any other suitable unit, for example, degree, centrad, etc.), and if also the angle γ between the edges of the prisms is given in degrees, the power p of the resultant prism may, therefore, be computed by the formula:

$$p = \sqrt{p_1^2 + p_2^2 + 2p_1 . p_2 . \cos\gamma} \ ,$$

and the angle θ which shows how the resultant prism is to be placed may be calculated by the formula:

$$\tan\theta = \frac{p_2 . \sin\gamma}{p_1 + p_2 . \cos\gamma}.$$

In particular, if $\gamma = 90°$, then $p = \sqrt{p_1^2 + p_2^2}$, $\tan\theta = \frac{p_2}{p_1}$.

As an illustration of the use of these formulæ, suppose that the deviations produced by the two prisms separately are 3° and 5°, and that the edges of the prisms are inclined to each other at an angle of 60°. Then $p_1 = 3°$, $p_2 = 5°$, $\gamma = 60°$, and hence the deviation produced by the two prisms together will be $p = \sqrt{9 + 25 + 15} = 7°$; and since $\tan\theta = \frac{5\sqrt{3}}{11}$, the resultant prism in this case is found to be a prism of power 7° placed with its edge at an angle of nearly 38° 13′ with that of the weaker of the two component prisms.

A "rotary prism" used for finding the necessary prismatic correction of a patient's eye is an instrument, circular in form, which consists of two ophthalmic prisms of equal power ($p_1 = p_2$) conveniently mounted so that the prisms can be rotated about an axis perpendicular to the plane of the instrument, one in front of the other, the angle between the prism-edges being shown by the positions of two marks which

move as the prisms are turned over a circular arc graduated in degrees. In the initial position when the two marks are at opposite ends of a diameter of the circular scale the base of one prism corresponds with the edge of the other, so that in this position the two prisms are equivalent to a glass plate with plane parallel faces ($\gamma = 180°$, $p = p_1 - p_2 = 0$). The maximum effect is obtained when the edges of the prism correspond ($\gamma = 0°$, $p = p_1 + p_2 = 2p_1$). With a device of this kind, we can obtain, therefore, any prismatic power from $p = 0$ to $p = 2p_1$.

On the other hand, we can resolve the effect of a given prism of power p into a component $p.\cos\theta$ in one direction and a component $p.\sin\theta$ in a direction perpendicular to the first. Thus, a prism of power 5 centrads with its edge at an angle of 30° to the horizontal is equivalent to a combination of two prisms of powers $\frac{5\sqrt{3}}{2}$ and $\frac{5}{2}$ centrads, with their edges horizontal and vertical, respectively.

PROBLEMS

1. Show how to construct the path of a ray refracted through a prism in a principal section; and prove the construction. Discuss the following special cases, and draw separate diagrams for each of them: (*a*) Incident ray normal to first face of prism, (*b*) Emergent ray "grazes" second face; (*c*) Ray traverses prism symmetrically; (*d*) Ray is incident on first face on side of normal towards the edge of the prism.

2. Show that the total deviation of a ray in a principal section of a prism of more highly refracting material than the surrounding medium is always away from the prism-edge. Discuss each of the three possible cases, viz., When the point where the two incidence-normals intersect falls (*a*) inside the prism, (*b*) outside the prism, and (*c*) on one of the two faces of the prism. Draw diagram for each case.

3. Obtain a formula for calculating the magnitude of the

angle of incidence at the first face of the prism of the ray which emerges from the prism along the second face; and discuss this formula for the cases when the refracting angle of the prism is (*a*) greater than 2A, (*b*) equal to 2A, (*c*) less than 2A but greater than A, (*d*) equal to A, and (*e*) less than A; where A denotes the so-called critical angle of the two media concerned. Draw diagram for each case.

4. Show that the deviation of a ray which goes symmetrically through a prism in a principal section is less than that of any other ray.

5. Show that the point of intersection of the incidence-normals to the two faces of a prism is equidistant from the incident ray and its corresponding emergent ray.

6. Construct the path of a ray refracted through a prism of small refracting angle; and show that the angle of deviation will also be a small angle of the same order of smallness, no matter how the ray falls on the prism.

7. What is the smallest angle that a glass prism ($n=1.5$) can have so that no ray can be transmitted through it? What is the magnitude of this angle for a water prism ($n=1.33$)? (Assume in each case that the prism is surrounded by air of index unity.)

Ans. 83° 37′ 14″; 97° 10′ 52″.

8. What must be the refracting angle of a prism whose index of refraction is equal to $\sqrt{2}$ in order that rays that are incident on one of its faces at angles less than 45° will be totally reflected at the other face? Ans. 75°.

9. The refracting angle of a prism is 60° and the index of refraction is equal to $\sqrt{2}$. Show that the angle of minimum deviation is 30°, and draw accurate diagram showing the construction of the path of this ray through the prism.

10. The refracting angle of a glass prism ($n=1.5$) is 60°, and the angle of incidence is 45°. Find the angle of deviation. What is the angle of minimum deviation for this prism?

Ans. 37° 22′ 52.5″; 37° 10′ 50″.

11. If the angle of minimum deviation of a ray traversing

a principal section of a prism is 90°, show that the index of refraction cannot be less than $\sqrt{2}$.

12. Find the angle of minimum deviation in the case of a glass prism ($n = 1.54$) of refracting angle 60°.

Ans. 40° 42′ 28″.

13. The minimum deviation for a prism of refracting angle 40° is found to be 32° 40′. Find the value of the index of refraction. Ans. 1.7323.

14. A glass prism of refracting angle 60° is adjusted so that the ray "grazes" the first face, and in this position the angle of emergence is found to be 29° 25′ 49″. Determine the index of refraction. Ans. 1.52.

15. A prism is made of glass of index 1.6, and the angle of minimum deviation is found to be 28° 31′ 20″. Calculate the refracting angle. Ans. 42° 39′ 44″.

16. The efracting angle of a water prism ($n = \frac{4}{3}$) is 30°. How must a ray be sent into this prism so that it will emerge along the second face?

Ans. Ray must lie on the side of the normal towards the edge of the prism, and make with the normal an angle of 25° 9′ 15″.

17. The angle of incidence for minimum deviation in the case of a prism of refracting angle 60° is 60°. Find the index of refraction. Ans. $\sqrt{3}$.

18. Find the index of refraction of a glass prism for sodium light for the following measurements: Refracting angle of prism = 45° 4′; angle of minimum deviation = 26° 40′.

Ans. 1.53.

19 The refracting angle of a prism is 30° and its index of refraction is 1.6. Find the angles of emergence and deviation for each of the following rays: (*a*) Ray meets first face normally; (*b*) Angle of incidence at first face is equal to 24° 28′; (*c*) Angle of incidence at first face is equal to 53° 8′; and (*d*) Ray "grazes" first face.

Ans. (*a*) 53° 8′; 23° 8′; (*b*) 24° 28′; 18° 56′; (*c*) 0°; 23° 8′; (*d*) 13° 59′; 46° 1′.

20. Find the refracting angle of a glass prism ($n = 1.52$) for which the minimum deviation is 15°. Ans. 27° 24′ 15″.

21. The refracting angle of a flint glass prism is measured and found to be 59° 56′ 22.4″; and the angles of minimum deviation for rays of light corresponding to the FRAUNHOFER lines D, F and H are also measured and found to have the following values: 46° 31′ 4.15″; 47° 35′ 59.2″; and 49° 30′ 5.7″, respectively. Calculate the values of the indices of refraction n_D, n_F, and n_H.

Ans. $n_D = 1.603528$; $n_F = 1.614771$; $n_H = 1.634183$.

22. The refracting angle of a crown glass prism is measured and found to be 60° 2′ 10.8″; and the angles of minimum deviation for rays of light corresponding to the FRAUNHOFER lines D, F and H are also measured and found to have the following values: 38° 38′ 14.3″; 39° 10′ 51.8″; and 40° 3′ 49.4″, respectively. Calculate the values of the indices of refraction n_D, n_F, and n_H.

Ans. $n_D = 1.516274$; $n_F = 1.522437$; $n_H = 1.532370$.

23. A prism is to be made of crown glass of index 1.526, and it is required to produce a minimum deviation of 17° 20′. To what angle must it be ground? Ans. 31° 20′.

24. A ray of light falls on one face of a prism in a direction perpendicular to the opposite face. Assuming that the refracting angle of the prism (β) is an acute angle, show that the ray will emerge along the opposite face if

$$\cot\beta = \cot A - 1,$$

where A denotes the critical angle of the prism-medium.

25. A ray "grazes" the first face of a prism and emerges at the second face in a direction perpendicular to the first face: show that the refracting angle (β) is such that

$$\cot\beta = \sqrt{n^2 - 1} - 1,$$

where n denotes the index of refraction of the prism-medium.

26. The refracting angle of a prism is 60° and the index of refraction is $\sqrt{7/3}$. What is the limiting angle of incidence of a ray that will be transmitted through the prism?

Ans. 30°.

27. Show that if ϵ_0 denotes the angle of minimum deviation of a prism of refracting angle β, the angle β cannot be greater than $(\pi - \epsilon_0)$ and the index of refraction cannot be less than $\sec\frac{\epsilon_0}{2}$.

28. Show that the minimum deviation of a prism of given index of refraction increases with increase of the refracting angle of the prism.

29. Derive the formula for the angle of deviation of a thin prism, and show that the deviation is approximately constant for all angles of incidence.

30. Show that when a thin glass prism of index $\frac{3}{2}$ is immersed in water of index $\frac{4}{3}$ the deviation of a ray will be only one-fourth of what it would be if the prism were surrounded by air.

31. The refracting angle of a prism of rock salt is 1° 30′. How much will a ray be deviated in passing through it? And what should be the refracting angle of a rock salt prism which is to produce a deviation of 48′? (Index of refraction of rock salt = 1.54.) Ans. 48′ 36″; 1° 29′.

32. What must be the refracting angle of a water prism of index $\frac{4}{3}$ to produce the same deviation as is obtained with a glass prism of index $\frac{3}{2}$ whose refracting angle is equal to 2°? Ans. 3°.

33. A glass prism of index 1.5 has a refracting angle of 2°. What is the power of the prism in prism-dioptries?

Ans. 1.745 prism-dioptries.

34. The power of a prism is 2 prism-dioptries and $n = 1.5$. Find the refracting angle. Ans. 2.29°.

35. A prism of refracting angle 1° 25′ bends a beam of light through an angle of 1° 15′. Calculate the index of refraction and the power of the prism in prism-dioptries.

Ans. $n = 1.882$; 2.18 prism-dioptries.

36. Two thin prisms are crossed with their edges at an angle of 30°. The first prism produces a deviation of 6° and the second a deviation of 8°. Find the deviation produced

by the single prism which is equivalent to this combination and the angle which the edge of the resultant prism must make with the edge of the first prism.

Ans. Deviation of resultant prism = 13.53°; angle between its edge and that of the 6°-prism = 17° 11′.

37. Two prisms, each of power 5 prism-dioptries, are combined base down with their base-apex lines inclined to the horizontal at angles of 45° and 135°. Find the equivalent single prism.

Ans. A prism of power a little more than 7 prism-dioptries, base down, vertical meridian (edge horizontal).

38. What will be the horizontal effect of a prism of power 10 placed with its base-apex line at an angle of 20° with the horizontal?

Ans. It will be the same as the effect of a prism of power nearly 9.4 in horizontal meridian (edge vertical).

39. The base-apex line of a prism of power 4 centrads makes an angle of 120° with the horizontal. Show that it is equivalent to a combination of two prisms, one of power 2 centrads in the vertical meridian (edge horizontal) and the other of power 3.46 centrads in the horizontal meridian (edge vertical).

40. Find the single prism equivalent to a combination of two prisms superposed with their base-apex lines at right angles to each other, the power of one being 3 and that of the other 4.

Ans. A prism of power 5 with its base-apex line inclined to that of the weaker prism at an angle of nearly 53° 8′.

41. Two equal prisms, each of power 3, are superposed in meridians inclined to each other at an angle of 120°. Find the equivalent single prism.

Ans. A prism of power 3 in a meridian halfway between the meridians of the two components.

42. The angle between the base-apex lines of a combination of two unit prisms is 82° 50′, and the bisector of this angle is horizontal. What is the horizontal effect of the combination? Ans. 1.5 units.

43. ABCDE is the principal section of a pentagonal prism. $AB=BC$, $AE=CD$, $\angle ABC=90°$, $\angle EAB=\angle BCD=112.5°$. A ray of light RS lying in the principal section is incident on the face BC at the point S. The ray enters the prism at this face, and is reflected, first, from the face AE, and then from the face DC, and emerges finally at a point P in the face AB in the direction PQ. Show that PQ makes a right angle with RS.

44. ABC is a principal section of a triangular prism, $\angle B=2\angle A$. A ray of light lying in the plane ABC is refracted into the prism at the side BC, and after undergoing two internal reflections, first, from the side AB and then from the side CA, emerges into the surrounding medium at the side AB. Show that the total deviation of the ray will be equal to the angle at B.

CHAPTER VI

REFLECTION AND REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE

63. Introduction. Definitions, Notation, etc.—The center of the spherical refracting or reflecting surface ZZ (Fig. 89) will be designated by C. The *axis* of the surface with respect to a given point M is the straight line joining M with C, and the point A where the straight line MC (produced if necessary) meets ZZ is called the pole or *vertex* of the surface with respect to the point M. Evidently, the spherical surface will be symmetrical around MC as axis, and the plane of the diagram which contains the axis is a meridian section of the surface.

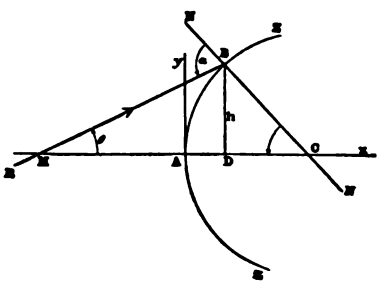


FIG. 89, *a*.—Ray incident on convex spherical surface crosses axis at point M in front of surface.

It will be convenient to take the vertex A as the *origin* of a system of plane rectangular coördinates; the axis of the surface being chosen as the x-axis and the tangent to the surface at its vertex, in the meridian plane of the diagram, being taken as the y-axis. *The positive direction of the x-axis is the direction of the incident ray which coincides with this line,* and since *the diagrams are all drawn on the supposition that the incident light goes from left to right,* a point lying on the x-axis to the right of A will be on the positive half of the axis. The positive direction of the y-axis is the direction found by rotating the positive half of the x-axis through a

right angle in a sense opposite to that of the motion of the hands of a clock in the meridian plane of the diagram. Accordingly, if the positive direction of the x-axis is along a

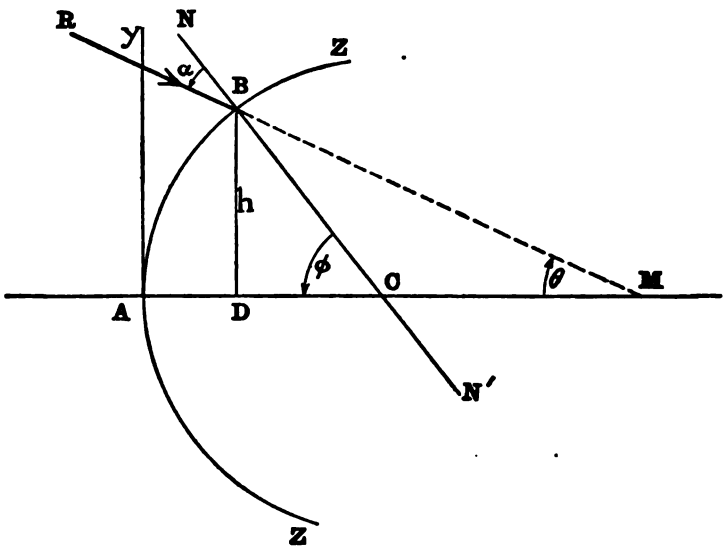


FIG. 89, *b*.—Ray incident on convex spherical surface crosses axis at point M on the other side of the surface.

horizontal line from left to right, the positive direction of the y-axis will be vertically upwards.

According as the center C lies on the same side of the spherical surface as that from which the incident light comes or on the opposite side, it is said to be *concave* (Fig. 89, *c* and *d*) or *convex* (Fig. 89, *a* and *b*), respectively. The radius r of the spherical surface is the abscissa of the center C, that is, $r = AC$. It is the step from A to C, and this is always a positive step for a convex surface (Fig. 89, *a* and *b*) and a negative step for a concave surface (Fig. 89, *c* and *d*). The radius of a convex surface whose center is 60 cm. from its vertex is $r = +60$ cm., and the radius of a concave surface of the same size is $r = -60$ cm.

It will be assumed in this chapter that any ray with which we are concerned lies in a meridian plane of the spherical surface; so that any straight line such as RB which represents the path of an incident ray will intersect the axis either "really" (Fig. 89, *a* and *c*) or "virtually" (Fig. 89, *b* and *d*) at some point designated here by M (see § 8). The point designated by R is any point on the incident ray RB at which the light arrives before it gets to either M or the incidence-point B. The straight line BC which joins the point of incidence with the center of the surface will be the incidence-normal, and if N designates a point on this normal lying in front of the spherical surface, then $\angle NBR = a$ will be the angle of incidence (§§ 13 & 27). The plane of this angle is the plane of incidence, which is the meridian plane of the diagram.

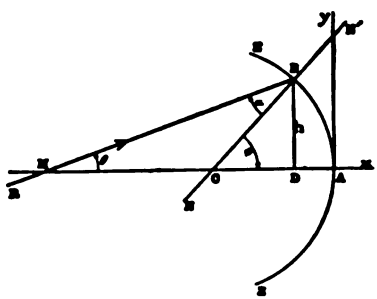


Fig. 89, *c*.—Ray incident on concave spherical surface crosses axis at point M in front of the surface.

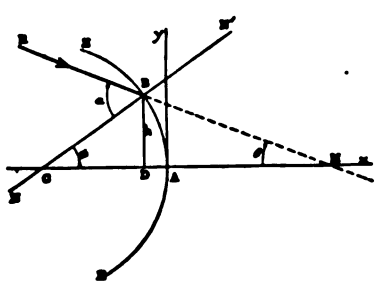


Fig. 89, *d*.—Ray incident on concave spherical surface crosses axis at point M on the other side of the surface.

From the incidence-point B draw BD perpendicular to the x-axis at D; the ordinate $h = DB$ is called the incidence-height of the ray. The *slope of the ray* is the acute angle through which the x-axis has to be turned around the point M in order that it may coincide in position (but not necessarily in direction) with the rectilinear path of the ray. If this angle is denoted by θ, then $\angle AMB = \theta$. Here, as always in the case of angular magnitudes (§ 13), counter-clockwise rotation

is to be reckoned as positive. And, finally, the acute angle at the center C of the spherical surface subtended by the arc BA will be denoted by ϕ. This angle, sometimes called the "central angle," is defined as the angle through which the radius CB must be turned around C in order to bring B into coincidence with the vertex A; thus, $\phi = \angle BCA$. The angles A, θ and ϕ, defined as above, are given by the following relations:

$$\tan\theta = -\frac{h}{DM}, \quad \sin\phi = \frac{h}{r}, \quad \alpha = \theta + \phi.$$

These formulæ should be verified for each of the diagrams Fig. 89, (*a*), (*b*), (*c*), (*d*).

Moreover, since $BM = \frac{DM}{\cos\theta}$, and since (see § 45)

$$DM = DC + CA + AM = r.\cos\phi - r + AM,$$

we find:

$$BM = \frac{r(\cos\phi - 1) + AM}{\cos\theta}.$$

Now in the special case *when the incidence-point* B *is very close to the vertex* A *of the spherical surface*, the angle of incidence α will be exceedingly small as will be also the angles denoted by θ and ϕ; and if these angles expressed in radians are all such small fractions that we may neglect their second and higher powers, so that in place of the sines (or tangents) we can write the angles themselves and put $\cos\theta = \cos\phi = \cos\alpha = 1$. Obviously, in such a case we shall have BM = AM. Under these circumstances the ray RB is called a *paraxial ray*, sometimes also a "central" or "zero" ray, $\alpha = \theta = \phi = 0$, approximately.

A paraxial ray is one whose path lies very near the axis of the spherical surface and which therefore meets this surface at a point close to the vertex and at nearly normal incidence: the angles denoted by α, θ *and* ϕ *being all so small that their second powers may be neglected.*

In this chapter and for several subsequent chapters we shall be concerned entirely with the procedure of paraxial

rays; that is, we shall consider only such rays as are comprised within a very narrow cylindrical region immediately surrounding the axis of the spherical surface which is likewise the axis of the cylinder. Accordingly, the only portion of the spherical surface that will be utilized for reflection or refraction will be a small zone whose summit is at A; so that, so far as paraxial rays are concerned, the rest of the spherical surface may be regarded as if it had no optical existence or at any rate as if it were opaque and non-reflecting. Thus, for example, the surface might be painted over with lampblack leaving bare and exposed only the small effective zone in the immediate vicinity of the vertex; or a screen might be set up at right angles to the axis close to the vertex with a small circular opening in it. Even then a source of light lying at a considerable distance off the axis would send rays which notwithstanding that they were incident near the vertex would not be paraxial rays.

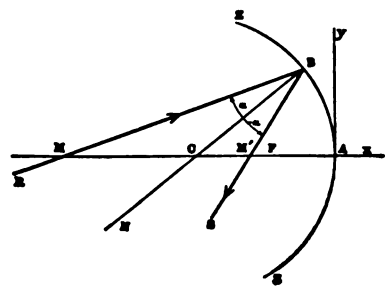


Fig. 90, *a*.—Reflection of ray at concave mirror.

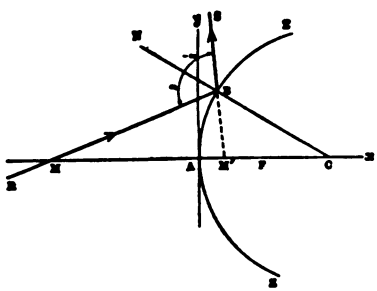


Fig. 90, *b*.—Reflection of ray at convex mirror.

64. Reflection of Paraxial Rays at a Spherical Mirror.—In the accompanying diagrams (Fig. 90, *a* and *b*) the straight line RB represents the path of an incident ray crossing the axis of a spherical mirror ZZ at the point M and incident on the mirror at the point B, and the straight line BS shows the path of the corresponding re-

flected ray crossing the axis, "really" (Fig. 90, *a*) or "virtually" (Fig. 90, *b*), at the point marked M′. By the law of reflection $\angle NBR = \angle SBN$ where BN is the incidence-normal and N designates a point on it which lies in front of the mirror. Since the normal bisects the interior or exterior angle at B of the triangle MBM′, the following proportion may be written:

$$\frac{CM}{BM} = \frac{M'C}{BM'}.$$

Now if the ray RB is a paraxial ray, the letter A may be substituted in the above equation in place of B, and thus * we obtain:

$$\frac{CM}{AM} = \frac{M'C}{AM'}.$$

Denoting the abscissae, with respect to the vertex A, of the axial points M, M′ by u, u', respectively, that is, putting $AM = u$, $AM' = u'$, and also, as stated in § 63, putting $AC = r$, we may write:

$$CM = CA + AM = -r + u = u - r,$$
$$M'C = M'A + AC = -u' + r = -(u' - r);$$

so that, introducing these symbols in the equation above, we obtain:

$$\frac{u-r}{u} = -\frac{u'-r}{u'}$$

which may be put in the form (see § 67):

$$\frac{1}{u} + \frac{1}{u'} = \frac{2}{r}.$$

If, therefore, the form and dimensions of the mirror are known (that is, if the value of r is assigned as to both magnitude and sign), and if also the position of the point M

* In writing this proportion, care must be taken to see that the two members of it shall have the same sign. For example, in each of the diagrams in Fig. 90, as they are drawn, the segments CM and AM have the same direction along the axis, so that for each of these figures the ratio CM : AM is positive. Now if the ratio M′C : AM′ is to be put equal to this ratio, it must be positive also, that is, the segments M′C and AM′ in each diagram must have the same direction.

where the incident paraxial ray crosses the axis of the spherical mirror is given, the abscissa u' of the point M′ where the corresponding reflected ray crosses the axis may be calculated by means of the expression:

$$u' = \frac{r.u}{2u - r}.$$

But the most noteworthy conclusion to be drawn from this formula is the fact that, provided the rays are paraxial, their actual slopes do not matter, for none of the angular magnitudes α, θ, or ϕ appears in the formula; which means that all paraxial rays which cross the axis at the point M before reflection will cross the axis after reflection in the spherical mirror at one and the same point M′. Thus, *a homocentric bundle of paraxial rays incident on a spherical mirror remains homocentric after reflection.* If, therefore, M designates the position of a luminous point in front of the mirror, and if the mirror is screened so that only such rays as proceed close to the axis are incident on it, the bundle of reflected rays will form at a point M′ on the straight line MC an ideal image of the luminous point M. According as the image-point M′ lies in front of the mirror (Fig. 90, *a*) or beyond it (Fig. 90, *b*), the image will be real or virtual, respectively. Thus, for a real image in a spherical mirror, the value of u' as found by the formula above will be negative, whereas for a virtual image it will be positive.

It may be noted also that the formula is symmetrical with respect to u and u', so that the equation will not be altered by interchanging the symbols u and u'; and hence it follows that *if* M′ *is the image of* M, *then likewise* M *may be regarded as the image of* M′. This is indeed merely an illustration of the general law known in optics as the "principle of the reversibility of the light-path" (§ 29). But the symmetry of the equation implies more than is involved in this principle; for it indicates that in the case of reflection object-space and image-space coincide completely, the actual paths of the incident and reflected rays both lying in the space in

front of the mirror. Accordingly, an incident ray and its corresponding reflected ray are always so related that when either is regarded as object-ray the other will be an image-ray.

THE DOUBLE RATIO OF FOUR POINTS ON A STRAIGHT LINE

65. Definition and Meaning of the Double Ratio.—It will be convenient and profitable at this place to turn aside from the special problem which is here under investigation in order to devote a few paragraphs to a brief explanation of the simpler metrical processes of modern projective geometry, which are of great utility in geometrical optics, especially when we are concerned with imagery by means of the so-called paraxial rays.

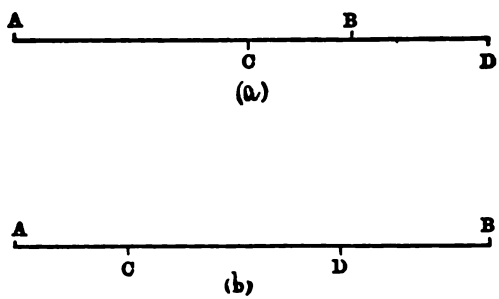


FIG. 91.—Line-segment AB divided (*a*) internally at C and externally at D, and (*b*) internally at C and D.

If L designates the position of a point on a straight line determined by the two points A, B, the line-segment AB is said to be divided at L in the ratio AL : BL. If the point L lies between A and B, the steps (see § 45) AL and BL are in opposite senses along the line, and the ratio AL : BL will be negative, and in this case we say that the segment AB is "divided internally" at L. On the other hand, if the point L does not lie between A and B, the ratio AL : BL will be positive, and we say that the segment AB is "divided externally" at L.

Accordingly, if A, B, C, D (Fig. 91, *a* and *b*) designate a

series of four points all ranged along a straight line in any order of sequence, the segment AB will be divided at C and D in the ratios AC : BC and AD : BD, respectively; and the quotient of these two ratios is called the *double ratio* (or "cross ratio") of the four points A, B, C, D. This double ratio is denoted symbolically by inclosing the four letters ABCD in parentheses; thus, according to the above definition,

$$(ABCD) = \frac{AC}{BC} : \frac{AD}{BD},$$

where the first two letters in the parentheses mark the endpoints of the segment and the last two letters designate the points of division. The line-segment CD is divided in the same way by the points A and B; for

$$(CDAB) = \frac{CA}{DA} : \frac{CB}{DB} = \frac{AC}{BC} : \frac{AD}{BD} = (ABCD).$$

According as the two ratios AC : BC and AD : BD have the same sign or opposite signs, the value of the double ratio (ABCD) will be positive or negative, respectively. Suppose, for example, that the segment AB is divided internally at C, as represented in both *a* and *b* of Fig. 91. Then the ratio AC : BC will be negative. Now if AB is divided also internally at D, as in Fig. 91, *a*, the ratio AD : DB will likewise be negative. Accordingly, if C and D are both points of internal division (or both points of external division), the double ratio (ABCD) will be positive. But if one of these points divides AB internally while the other divides it externally (Fig. 91, *b*), the double ratio (ABCD) will be negative.

In order to form a clear idea of the values which (ABCD) may assume, let us suppose that the points designated by A, B and C in Fig. 92 represent three stationary points on a straight line x, and that O designates another fixed point not on this line. The straight line x and the point O together determine a plane which is the plane of the diagram. Now let y designate a second straight line lying in this plane and

passing through O, and let the point of intersection of the straight lines x and y be designated by Y. And if the straight line y is supposed to turn around O as a pivot in a sense, say, opposite to that of the motion of the hands of a clock, the point Y will be a variable point moving along the straight line x constantly in the same sense, namely, in Fig. 92 from left to right. Assume, for example, that the three stationary points A, C, B are ranged along the straight line x from left to right in the order named, as shown in the figure; and suppose that the variable point Y starts originally at B, so that the revolving line OY or y coincides initially with the "ray" marked b in the figure and $BY = BB = 0$, and, consequently, the ratio $AY : BY = \infty$, Hence, under these circumstances the initial value of the double ratio of the four points A, B, C, Y will be:

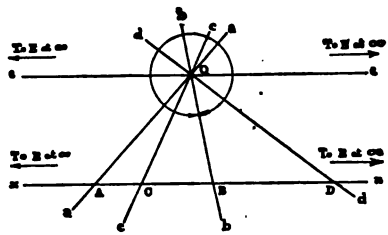


FIG. 92.—Central projection from O of the point-range ABCDE lying on the straight line x.

$$(ABCY) = \frac{AC}{BC} : \frac{AY}{BY} = 0.$$

When the revolving ray has turned through $\angle BOD$, where D designates a point lying on the straight line x to the right beyond B, the point Y will be at D outside the segment AB and the double ratio (ABCY) will be negative, as explained above. As y continues to revolve around O, the point Y will move farther and farther to the right along the straight line x, until when y is parallel to x, and in the position of the ray marked e in the figure, the point Y will then coincide with *the infinitely distant point* E *of the straight line* x. Now $AE = BE = \infty$, and hence $AE : BE = 1$; and therefore when Y is at E,

$$(ABCY) = \frac{AC}{BC} : \frac{AE}{BE} = \frac{AC}{BC}.$$

When the revolving ray y has turned beyond the position represented by the straight line e, the point Y which had just vanished at one end E of the straight line x now reappears from the other end E, proceeding along it still in the same sense from left to right. Thus, before the ray y has executed a complete revolution, the point Y will pass through A, and at this moment, $AY = AA = 0$, and

$$(ABCY) = \frac{AC}{BC} : \frac{AY}{BY} = \frac{AC}{BC} : \frac{AA}{BA} = -\infty ;$$

and thus we see that as the point Y has traversed the straight line x from B *via* the infinitely distant point E to A, the double ratio (ABCY) has assumed all negative values from 0 to $-\infty$. Finally, as the ray y completes its revolution by turning from the position a to its initial position b, the point Y moves from A *via* C to B. When Y is at C, $AY = AC$, $BY = BC$, and

$$(ABCY) = \frac{AC}{BC} : \frac{AY}{BY} = +1 ;$$

so that in passing along x from A to C, (ABCY) assumes all positive values comprised between $+\infty$ and $+1$. Between C and B, it has all positive values less than unity. Thus, as the point Y traverses the straight line x continually in the same sense until it has returned to its starting point, the double ratio (ABCY) will assume all possible values both positive and negative.

In general, since

$$(ABCD) = \frac{AC}{BC} : \frac{AD}{BD} = \frac{BD}{AD} : \frac{BC}{AC} = \frac{CA}{DA} : \frac{CB}{DB} = \frac{DB}{CB} : \frac{DA}{CA},$$

we may write:

$$(ABCD) = (BADC) = (CDAB) = (DCBA).$$

66. Perspective Ranges of Points.—If A, B, C, etc., designate the positions of the points of a point-range x (Fig. 92) these points are said to be "projected" from a point O outside of x by the straight lines or "rays" OA, OB, OC, etc.; and if these rays intersect another straight line x' (Fig. 93) in the points A′, B′, C′, etc., the two point-ranges x, x' are said to be in *perspective* with respect to the point O as *center*

of perspective. The points A, A′; B, B′; C, C′; etc., are called pairs of corresponding points of the two perspective point-ranges x, x'.

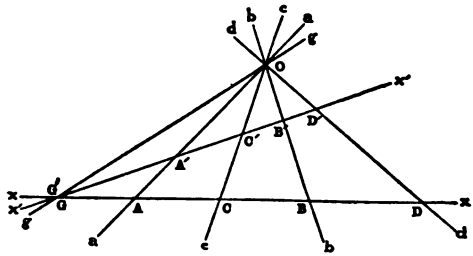


FIG. 93.—The point-ranges ABCD and A′B′C′D′ are in perspective relation with respect to the point O as centre of perspective.

If A, B, C, D designate the positions of any four points of x, and if A′, B′, C′, D′ designate the corresponding points on x', then

$$(A'B'C'D') = (ABCD),$$

as we shall proceed to show.

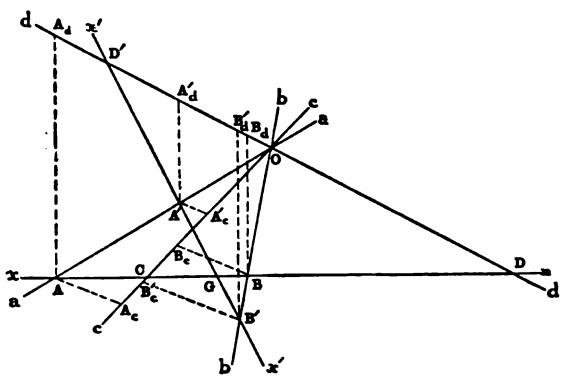


FIG. 94.—Straight lines x, x' are bases of two point-ranges in perspective, so that (ABCD) = (A′B′C′D′).

Through the points A, B, A′ and B′ (Fig. 94) draw four parallel lines AA_c, BB_c, $A'A_c'$ and $B'B_c'$ meeting the ray OC

or c in the points A_c, B_c, A_c' and B_c', respectively; and through these same points draw four other parallel lines AA_d, BB_d, $A'A_d'$ and $B'B_d'$ meeting the ray OD or d in the points A_d, B_d, A_d' and B_d', respectively. Then, evidently,

$$\frac{AC}{BC}=\frac{AA_c}{BB_c},\quad \frac{AD}{BD}=\frac{AA_d}{BB_d},$$
$$\frac{A'C'}{B'C'}=\frac{A'A_c'}{B'B_c'},\quad \frac{A'D'}{B'D'}=\frac{A'A_d'}{B'B_d'};$$

hence,

$$(ABCD)=\frac{AC}{BC}:\frac{AD}{BD}=\frac{AA_c}{BB_c}:\frac{AA_d}{BB_d},$$
$$(A'B'C'D')=\frac{A'C'}{B'C'}:\frac{A'D'}{B'D'}=\frac{A'A_c'}{B'B_c'}:\frac{A'A_d'}{B'B_d'}.$$

Now

$$\frac{A\,A_c}{A'A_c'}=\frac{A\,A_d}{A'A_d'},\quad \frac{B\,B_c}{B'B_c'}=\frac{B\,B_d}{B'B_d'},$$

and, consequently,

$$(A'B'C'D')=(ABCD),$$

as was to be proved.

67. The Harmonic Range.—The special case when the points C and D divide the line-segment AB internally and externally in the same numerical ratio, so that

$$\frac{AC}{BC}=-\frac{AD}{BD},$$

demands attention, particularly because it is a case that we shall meet again in the theory of the reflection of paraxial rays at a curved mirror. Under these circumstances, the value of the double ratio is

$$(ABCD)=-1;$$

and then we say that *the segment AB is divided harmonically at* C *and* D, or also the segment CD is divided harmonically at A and B. For example, the perpendicular bisectors of the exterior and interior angles of a triangle divide the opposite side of the triangle harmonically in the ratio of the other two sides.

Four harmonic points may be defined not merely by the metrical relation that their double ratio is equal to -1, but also by a geometrical relation, as we shall now show.

Let P, Q, R, S (Figs. 95 and 96) designate the positions of four points lying all in one plane, no three of which are in the same straight line. These four points will determine six straight lines, viz., PQ, PR, PS, QR, QS, and RS, which are called the *sides* of the *complete quadrilateral* whose four vertices are at the points P, Q, R, and S. Any two of these lines which together contain all the vertices form a pair of opposite sides of the quadrilateral. Accordingly, there are three pairs of opposite sides, viz., PQ and RS which meet in a point designated by A, PS and QR which meet in a point designated by B, and QS and PR which meet in a point designated by O. The three points A, B and O are sometimes called the *secondary vertices* of the quadrilateral. We shall explain now what connection this figure has with a harmonic range of points.

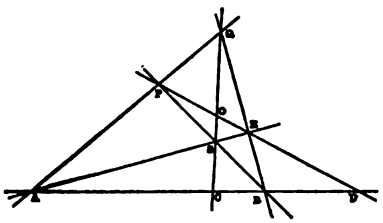


Fig. 95.—Complete quadrilateral PQRS; (ABCD) = −1.

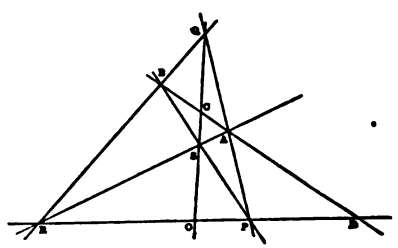


Fig. 96.—Complete quadrilateral PQRS; (ABCD) = −1

The secondary vertices A and B are determined by the two pairs of opposite sides PQ, RS and PS, QR; and the points C and D where the third pair of opposite sides QS and PR meet the straight line AB divide the segment AB harmonically. For, since A, B, C, D and P, R, O, D are in

perspective relation with respect to the point Q as center of perspective (§ 66), therefore

$$(ABCD) = (PROD).$$

But P, R, O, D and B, A, C, D are also in perspective to each other with respect to the point S as center of perspective; consequently,

$$(PROD) = (BACD).$$

It follows therefore that

$$(ABCD) = (BACD).$$

But by the definition of the double ratio

$$(BACD) = \frac{1}{(ABCD)}.$$

Accordingly, here we must have:

$$(ABCD) = \frac{1}{(ABCD)},$$

or

$$(ABCD)^2 = 1.$$

According to this equation, therefore, the double ratio (ABCD) must be equal to $+1$ or -1. But we saw above (§ 65) that the double ratio of four points A, B, C, D in a straight line can be equal to $+1$ only in case one of the points A, B is coincident with one of the pair C, D; which cannot happen in case of the four points A, B, C, D of the quadrilateral PQRS. Therefore, we must have here:

$$(ABCD) = -1;$$

and hence, by definition, the points A, B are harmonically separated by the points C, D. Similarly, also, the points P, R are harmonically separated by the points O, D.

If A, B, C, D is a harmonic range of points, then

$$\frac{BC}{AC} = \frac{DB}{AD}, \text{ or } \frac{BA+AC}{AC} = \frac{DA+AB}{AD},$$

that is,

$$\frac{AC-AB}{AC} = \frac{AB-AD}{AD};$$

which may finally be written in the form:

$$\frac{1}{AC}+\frac{1}{AD}=\frac{2}{AB};$$

an equation that is characteristic of a harmonic range of four points A, B, C, D (*cf.* § 64).

68. Application to the Case of the Reflection of Paraxial Rays at a Spherical Mirror.—When paraxial rays are reflected at a spherical mirror whose center is at C, we saw (§ 64) that $CM : AM = M'C : AM'$, where M, M′ designate the positions of a pair of conjugate points lying on a central ray which crosses the mirror at the point marked A (Fig. 90, *a* and *b*); and therefore

$$\frac{CM}{CM'}:\frac{AM}{AM'}=-1 \text{ or } (CAMM')=-1.$$

Consequently, the four points C, A, M, M′ are a harmonic range of points lying on the central ray AC, and we may say that the pair of conjugate points M, M′ is harmonically separated by the center of the mirror C and the point A where the central ray meets the mirror. Thus, if we know the positions of three of these points, we can construct the position of the fourth point by the aid of the properties of the complete quadrilateral (§ 67). For example, the image-point M′ conjugate to a given point M with respect to a spherical mirror may be constructed as follows:

Draw a straight line x (Fig. 97, *a* and *b*) to represent the axis of the mirror, and mark on it the positions of the three given points, A, C and M, which may be ranged along this line in any sequence whatever depending on the form of the mirror and on whether the object-point M is real or virtual. Through M draw another straight line in any convenient direction, and mark on it two points which we shall call Q and S, and draw the straight lines AQ and CS meeting in a point R and the straight lines AS and CQ meeting in a point P. Then the straight line PR will intersect the straight line x in the point M′ which is conjugate to M with respect to a spherical mirror whose vertex is at A and whose center

is at C. It will be remarked that in performing this construction the only drawing instrument that is needed is a straight-edge.

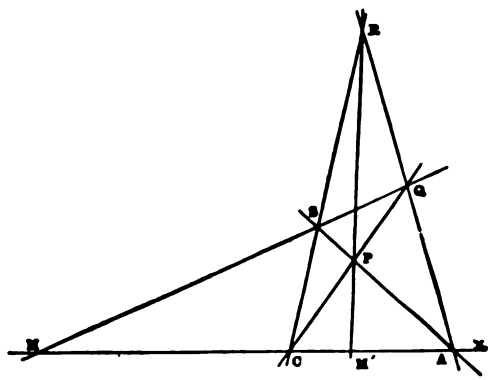


Fig. 97, *a*.—Concave Mirror: Construction of point M′ conjugate to axial point M in front of the mirror.

If the mirror is concave, the possible sequences of these four points on the axis are M, C, M′, A; M′, C, M, A; and C, M, A, M′, when the object-point M is real, and C, M′,

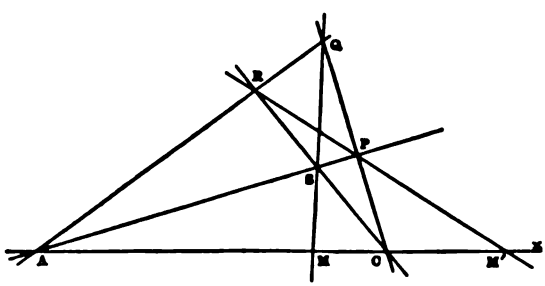


Fig. 97, *b*.—Convex Mirror: Construction of point M′ conjugate to virtual object-point M on axis of mirror.

A, M, when the object-point M is virtual. In the case of a convex mirror the points may occur in any one of the following arrangements: M, A, M′, C, when the object-point M

is real, and M′, A, M, C; A, M, C, M′ and A, M′, C, M, when the object-point M is virtual. The student should satisfy himself as to the accuracy of these statements by drawing a diagram for each of these eight sequences according to the directions for the construction as given above. Fig. 97, *a* shows the case of a concave mirror with the points in the order M, C, M′, A; whereas Fig. 97, *b* represents a convex mirror with a virtual object-point at M, the order in this case being A, M, C, M′.

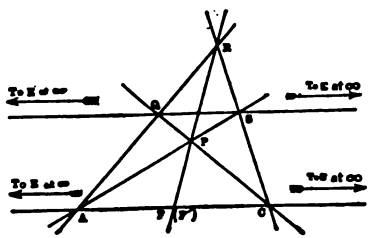


FIG. 98, *a*.—Focal point of convex mirror (AF = FC)

69. Focal Point and Focal Length of a Spherical Mirror.—In the special case when the object-point M coincides with the infinitely distant point *E* of the *x*-axis, the conjugate point M′ will lie at a point F′ (Fig. 98, *a* and *b*) determined by the relation:

$$(CAEF') = -1,$$

and since here $CE = AE = \infty$, we must have:

$$AF' = F'C.$$

This means that a cylindrical bundle of incident paraxial rays parallel to the axis of a spherical mirror will be transformed into a conical bundle of reflected rays with its vertex at a point F′ which is midway between the vertex A and the center C.

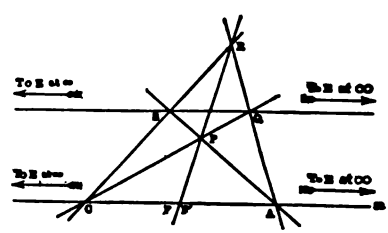


FIG. 98, *b*.—Focal point of concave mirror (AF = FC)

If, on the other hand, the image-point M′ coincides with the infinitely distant point E, the conjugate object-point M will lie on the axis at a point F determined by the relation:

$$(CAFE') = -1,$$

and therefore we obtain here in the same way as above:

$$AF = FC.$$

Accordingly, a conical bundle of incident rays with its vertex at a point F midway between the vertex of the mirror and its center will be transformed into a cylindrical bundle of reflected rays parallel to the axis of the mirror. The letters F and F′ will be used to designate the positions of the so-called *focal points* of an optical system which is symmetric around an axis. They are not a pair of conjugate points, as might naturally be inferred from the fact that they are designated by the same letter. In the case of a spherical mirror these two points, as we have seen, are coincident with each other, which is a consequence of the identity of object-space and image-space to which reference was made at the conclusion of § 64. The focal point of a concave mirror lies in front of the mirror, as shown in Fig. 98, *b*, so that paraxial rays parallel to the axis will be reflected at a concave mirror to a real focus at F; whereas in the case of a convex mirror the focal point F lies behind the mirror (virtual focus), as shown in Fig. 98, *a*.

The *focal length* f of a spherical mirror may be defined as the abscissa of the vertex A with respect to the focal point F as origin; that is, $f = FA$. Hence, *according as the mirror is concave or convex, the focal length will be positive or negative, respectively.* It may be remarked that the signs of f and r are always opposite, the relation between these magnitudes being given by the following formula:

$$f = -\frac{r}{2} \text{ or } r = -2f.$$

Hence, also, the abscissa-relation obtained in § 64 may be written in terms of f instead of r as follows:

$$\frac{1}{u}+\frac{1}{u'}+\frac{1}{f}=0;$$

where, however, it must be borne in mind that, whereas the abscissæ u, u' are measured from the vertex A as origin, the focal length f is measured from the focal point F.

If the abscissæ, with respect to the focal point F, of the pair of conjugate axial points M, M′ are denoted by x, x', that is, if $FM = x$, $FM' = x'$, then, since

$$AM = AF + FM, \qquad AM' = AF + FM',$$

the connection between the u's and the x's is given by the following equations:

$$u = x - f, \qquad u' = x' - f;$$

and substituting these values in the formula above and clearing of fractions, we derive the so-called *Newtonian formula*, viz.:

$$x \cdot x' = f^2;$$

which is an exceedingly simple and convenient form of the abscissa-relation between a pair of conjugate axial points. The right-hand side of this equation is essentially positive, and hence the abscissæ x, x' must always have like signs. Consequently, *in a spherical mirror the conjugate axial points* M, M′ *lie always both on the same side of the focal point* F.

70. Graphical Method of exhibiting the Imagery by Paraxial Rays.—The points M, M′ in Fig. 99, *a* and *b* desig-

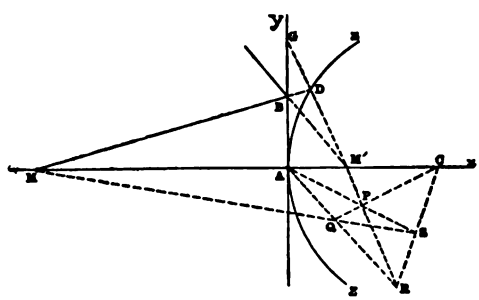


Fig. 99, *a*.—For paraxial rays the reflecting (or refracting) surface must be represented in diagram by the straight line A*y*, not by the curved line AZ.

nate the positions on the axis of a spherical mirror of a pair of conjugate points constructed according to the method explained in § 68. On the reflecting sphere ZZ take a point D, and draw the straight lines MD, M′D meeting the tan-

gent Ay in the plane of these lines in the points B, G, respectively. Also, draw the straight line M'B. Now if the point D were very close to the vertex A of the mirror, then the straight line MD would represent the path of an incident

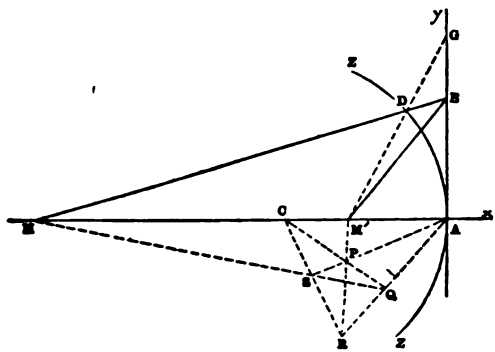


FIG. 99, *b*.—For paraxial rays the reflecting (or refracting) surface must be represented in diagram by the straight line Ay, not by the curved line AZ.

paraxial ray crossing the axis at M, and the path of the corresponding reflected ray would be along the straight line DM'. But under these circumstances, the three points designated here by the letters D, B, G would all be so near together that even when we cannot regard D as absolutely coincident with A, we may consider D, B and G as all coincident with one another. Therefore, when the ray is paraxial, we may, and, in fact, in the diagram we *must*, regard the straight line BM' as showing the path of the reflected ray. It is quite essential that this point which is seldom clearly explained should be rightly apprehended by the student. In diagrams showing the imagery by means of paraxial rays the duty of the straight lines that are drawn is not primarily to represent the actual paths of the rays themselves but to locate by their intersections the correct positions of the pairs of corresponding points in the object-space and image-space.

In the construction of such diagrams, a practical difficulty

is encountered due to the fact that, whereas in reality paraxial rays are comprised within the very narrow cylindrical region immediately surrounding the axis of the spherical surface (§ 63), it is obviously quite impossible to show them this way in the figure, because it would be necessary to take the dimensions of the drawing at right angles to the axis so small that magnitudes of the second order of smallness would no longer be perceptible at all; thus, for example, the points B, D, G in Fig. 99 would have to be shown as one point. On the other hand, if the lines in the diagram are not all drawn close to the axis, the relations which have been found above will cease to be applicable, so that, for instance, the rays shown in such a drawing would not intersect in the places demanded by the formulæ.

Accordingly, in order to overcome this difficulty, a method of constructing these figures has been very generally adopted, which, although it is confessedly in the nature of a compromise, has been found to be on the whole quite satisfactory, and wherein at any rate the geometrical relations are in agreement with the algebraic conditions, which is the essential requirement. In this plan, while the dimensions parallel to the axis remain absolutely unaltered, the dimensions at right angles to the axis are all prodigiously magnified in the same proportion. Thus, for example, if the incidence-height $h = \mathrm{DB}$ (Fig. 89) is a small magnitude of the order, say, of one-thousandth of the unit of length, it will be shown in the figure magnified a thousand times; whereas another ordinate whose height was only one one-millionth of the unit of length and which, therefore, would be of the second order of smallness as compared with h, would appear even in the magnified diagram as a magnitude of the first order of smallness. And if the ordinate denoted by h, although in reality infinitely small, is represented in the drawing by a line of finite length, an ordinate of the second order of smallness as compared with h will be entirely unapparent in the magnified diagram.

Of course, as already intimated, one effect of this lateral enlargement will be to misrepresent to some extent the relations of the lines and angles in the figure. For instance, the circle in which the spherical mirror (or refracting surface) is cut by the plane of a meridian section will thereby be transformed into an infinitely elongated ellipse with its major axis perpendicular to the axis of the spherical surface, and this ellipse will appear in the diagram as a straight line Ay tangent to the circle at A. The minor axis of the ellipse remains unchanged and equal to the diameter $2r$ of the circle, and moreover the center of the ellipse remains at the center C of the circle. But the most apparent change will be in the angular magnitudes which will be completely altered and distorted. For example, every straight line drawn through the center C really meets the circle ZZ (Fig. 89) normally, but in the distorted figure the axis of symmetry will be the only one of such lines which will be perpendicular to the straight line Ay which takes the place of the circular arc ZZ. Angles which in reality are equal will appear unequal, and *vice versa*. However—and after all this is the really essential matter—*the absolute dimensions of the abscissæ and the relative dimensions of the ordinates will not be changed at all;* and therefore lines which are really straight will appear as straight lines in the figure, and straight lines which are parallel will be shown as such. The abscissa of the point of intersection of a pair of straight lines in the drawing will be the true abscissa of this point.

In such a diagram, therefore, any ray, no matter what slope it may have nor how far it may be from the axis, is to be considered as a paraxial ray. The meridian section of the spherical reflecting or refracting surface *must* be represented in the figure by the straight line Ay (y-axis), and the position of the center C with respect to the vertex A will show whether the surface is convex or concave.

71. Extra-Axial Conjugate Points.—If we suppose that the axis of the spherical mirror is rotated about the center C

through a small angle ACU, so that the vertex A moves along the mirror to a neighboring point U, the conjugate axial points M, M′ will describe also small arcs MQ, M′Q′ of concentric circles; and, evidently, the points Q, Q′ will be

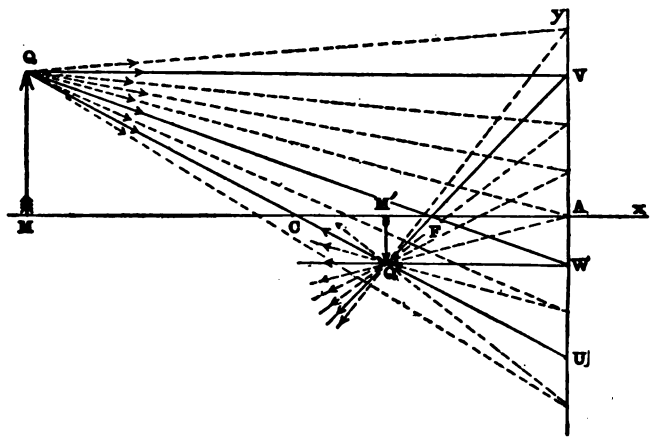


FIG. 100.—Concave mirror: Object is a small line MQ perpendicular to axis; its image M′Q′ is real and inverted.

harmonically separated (§§ 67, 68) by the points C, U, so that (CUQQ′) = (CAMM′) = − 1. Thus, we see how the point Q′ is the image-point conjugate to the extra-axial object-point Q. In the diagram (Fig. 100) the circular arcs AU, MQ and M′Q′ will appear as straight lines perpendicular to the axis, as explained in § 70. We derive, therefore, without difficulty the following conclusions:

(1) *Th image, in a spherical mirror, of a plane object perpendicular to the axis is likewise a plane perpendicular to the axis;* (2) *A straight line passing through the center of the spherical mirror intersects a pair of such conjugate planes in a pair of conjugate points;* and (3) *To a homocentric bundle of incident paraxial rays proceeding from a point* Q *in a plane perpendicular to the axis of a spherical mirror there corre-*

sponds a homocentric bundle of reflected rays with its vertex Q' lying in the conjugate image-plane.

In order to *construct the image-point Q' of the extra-axial object-point* Q, we have merely to find the point of intersection after reflection at the spherical mirror of any two

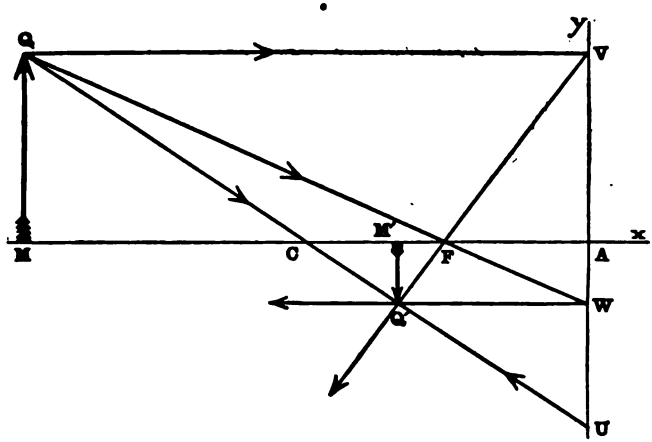


FIG. 101, *a*.—Lateral magnification and construction of image in concave mirror.

rays emanating originally from Q. The diagrams (Fig. 101, *a* and *b*), which are drawn according to the method explained in § 70, exhibit this construction for the cases when the mirror is concave and convex. Of the incident rays proceeding from Q, it is convenient to select for this purpose two of the following three, namely: the ray QC which proceeding towards the center C meets the spherical mirror normally at U, whence it is reflected back along the same path; the ray QV which proceeding parallel to the axis and meeting the mirror in the point designated by V is reflected at V along the straight line joining V with the focal point F; and the ray QW which being directed towards the focal point F is reflected at W in a direction parallel to the axis. The point where these reflected rays intersect will be the image-

point Q′. Moreover, having located the position of Q′, we can draw QM, Q′M′ perpendicular to the axis at M, M′, respectively; and then M′Q′ will be the image of the small object-line MQ. In Fig. 101, *a* the image M′Q′ is real and inverted, whereas in Fig. 101, *b* it is virtual and erect.

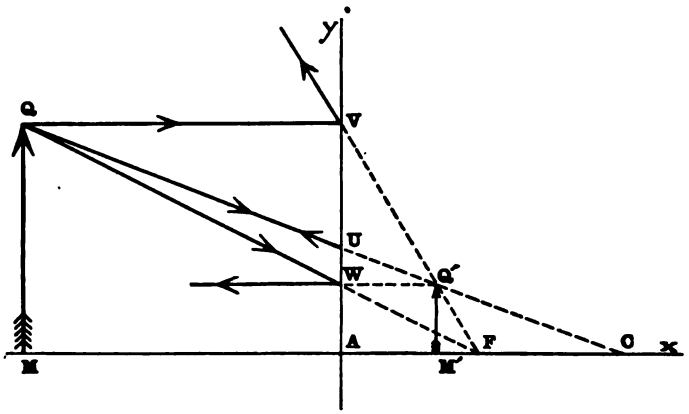


FIG. 101, *b*.—Lateral magnification and construction of image in convex mirror.

Whether the image is real or virtual and erect or inverted will depend both on the position of the object and on the form of the mirror.

If the object-point Q is supposed to move, say, from left to right along the straight line QV drawn parallel to the axis of the mirror, the corresponding image-point Q′ will traverse the straight line VF continuously in the same direction. Thus, in the diagrams (Fig. 102, *a* & *b*) the numerals 1, 2, 3, etc., ranged in order from left to right along a straight line parallel to the axis of the mirror, show a number of successive positions of the object-point, while the primed numbers 1′, 2′, 3′, etc., lying along the straight line VF, show the corresponding positions of the image-point. The straight lines 11′, 22′, 33′, etc., all meet at the center C of the mirror.

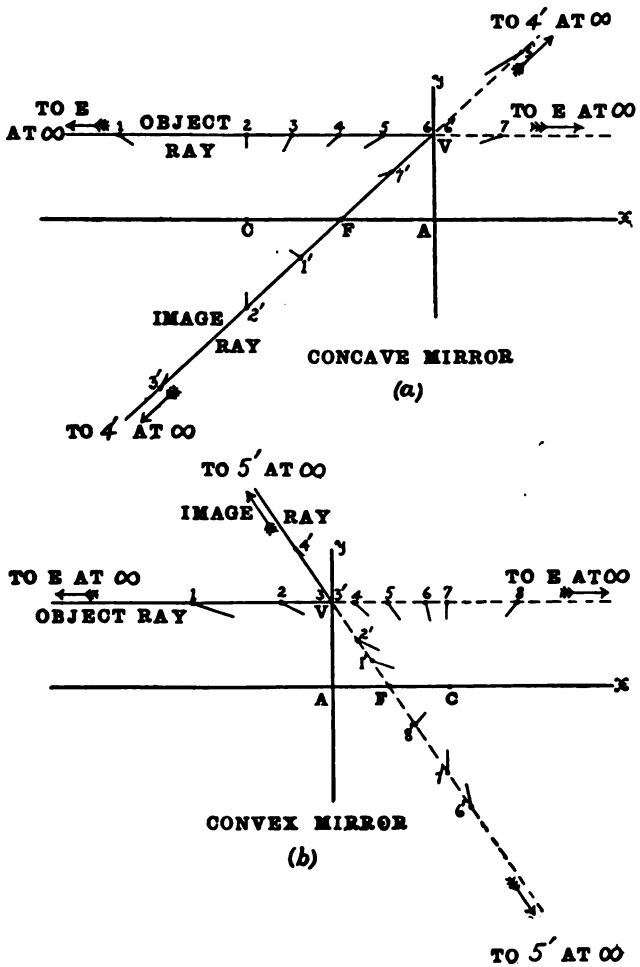


FIG. 102, *a* and *b*.—Imagery in (*a*) concave mirror, (*b*) convex mirror.

72. The Lateral Magnification.—If the ordinates of the pair of extra-axial conjugate points Q, Q′ are denoted by y, y', respectively, that is, if in Fig. 101, *a* and *b*, $MQ=y$, $M'Q'=y'$, the ratio y'/y is called the *lateral magnification* at the axial point M. This ratio will be denoted by μ; thus, $\mu=y'/y$. The sign of this function μ indicates whether the image is erect or inverted. The lateral magnification may have any value positive or negative depending only on the position of the object.

In the similar triangles MCQ, M′CQ′

$$M'Q' : MQ = M'C : MC;$$

and since

$$M'C = r-u', \qquad MC = r-u,$$

where $u=AM$, $u'=AM'$, $r=AC$; and since according to the abscissa-formula (§ 64)

$$\frac{r-u'}{r-u}=-\frac{u'}{u},$$

we derive the following formula for the lateral magnification in the case of a spherical mirror:

$$\mu=\frac{y'}{y}=-\frac{u'}{u}.$$

Also, from the figure we see that

$$\frac{M'Q'}{MQ}=\frac{AW}{MQ}=\frac{FA}{FM}=\frac{M'Q'}{AV}=\frac{FM'}{FA};$$

and since $FM=x$, $FM'=x'$, and $FA=f$, we derive also another formula for the lateral magnification, as follows:

$$\mu=\frac{y'}{y}=\frac{f}{x}=\frac{x'}{f}.$$

This expression shows that the lateral magnification is inversely proportional to the distance of the object from the focal plane.

73. Field of View of a Spherical Mirror.—When the image of a luminous object is viewed in a spherical mirror, the axis of the mirror is determined by the straight line O′C

(Fig. 103, *a* and *b*) joining the center O′ of the pupil of the observer's eye with the center C of the mirror; and, on the assumption that the image is formed by the reflection of paraxial rays, the actual portion of the mirror that is utilized

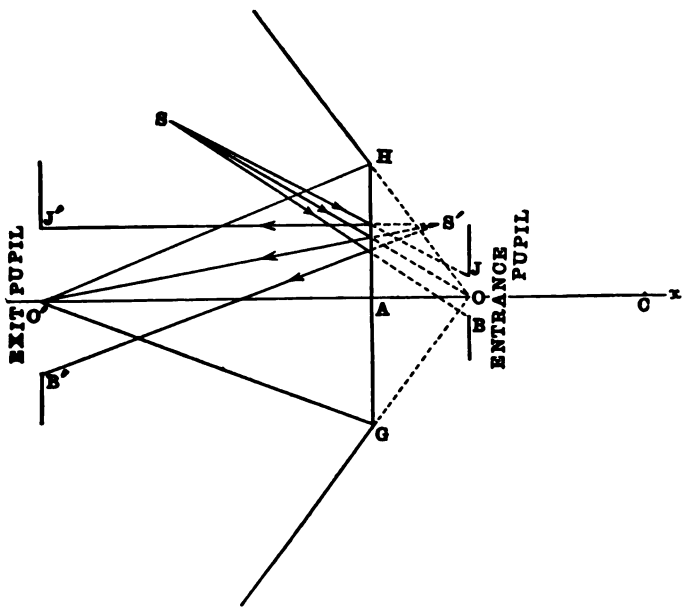


FIG. 103, *a*.—Field of view for eye in front of convex mirror.

consists of a small circular zone immediately surrounding the vertex A where the axis meets the reflecting surface. According to the method of drawing these diagrams which was described in § 70, the line-segment GH which is perpendicular to the axis at A and which is bisected at A will represent a meridian section of this zone in the plane of the figure, so that the points designated by G, H are opposite extremities of a diameter of the effective portion of the mirror.

All the reflected rays that enter the eye at O′ must necessarily lie within the conical region determined by revolving the isosceles triangle O′GH around the axis of the mirror.

The outermost rays that can possibly be reflected into the eye at O′ will be the rays that are reflected along the straight lines HO′ and GO′. In order to see a real image in a concave mirror (Fig. 103, *b*), the eye must be placed in front of the

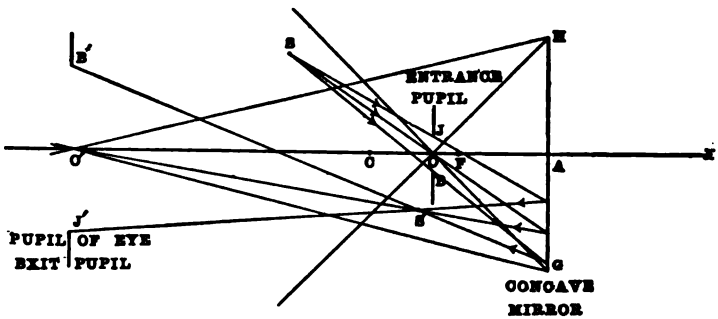


FIG. 103, *b*.—Field of view for eye in front of concave mirror.

mirror at a distance greater than the length of the radius. The incident rays corresponding to the extreme reflected rays will intersect in a point O which is conjugate to O′; and hence the *field of view* (§ 9) within which all object-points must lie in order that their images in the mirror may be visible to an eye at O′ will be limited by the surface of a right circular cone generated by the revolution of the isosceles triangle OHG around the axis of the mirror. Thus, exactly as in the case of the corresponding problem in connection with the field of view of a plane mirror (§ 16), the contour of the effective portion of the spherical mirror acts also as a *field-stop* for the imagery produced by paraxial rays.

Through O′ draw B′J′ at right angles to the axis of the mirror, and mark the points B′, J′ at equal distances from O′ on opposite sides of the axis. Then B′J′ may be supposed to represent the diameter in the plane of the diagram of the iris opening of the pupil of the observer's eye. Construct by the method described in § 71, the object-line BJ whose image in the mirror is B′J′. Evidently, any ray which after

reflection enters the pupil of the eye between B′ and J′ must before reflection have passed, really or virtually, through the conjugate point on the straight line between B and J. In fact, the circle described around O as center in the transversal plane perpendicular to the axis at O with radius OB will act like a material stop to limit the apertures of the bundles of incident rays. It is the so-called *entrance-pupil* of the system, while the pupil of the eye plays the part of the *exit-pupil* (see § 16). Thus, for example, if S designates the position of a luminous point lying anywhere within the field of view, the eye at O′ will see the image of S at S′ by means of a bundle of rays which are drawn from S to all points of the entrance-pupil and which after reflection at the mirror are comprised within the cone which has its vertex at S′ and the exit-pupil as base. The entrance-pupil BJ is the aperture-stop of the system (§ 11).

74. Refraction of Paraxial Rays at a Spherical Surface.— In the accompanying diagrams Fig. 104, *a* and *b*, the straight line RB represents an incident ray meeting the spherical refracting surface ZZ at B, while the straight line BS shows

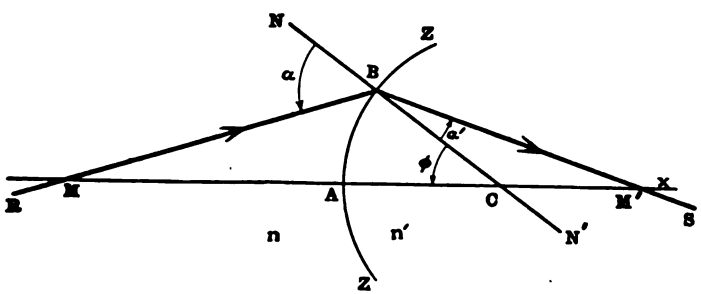


Fig. 104, *a*.—Convex spherical refracting surface ($n' > n$).

the path of the corresponding refracted ray. If the position of the point M where the incident ray crosses the axis is given, the problem is to determine the position of the point M′ where the refracted ray meets the axis. The angles of

incidence and refraction are $\angle NBR = \alpha$, $\angle N'BS = \alpha'$, and by the law of refraction:

$$n' . \sin\alpha' = n . \sin\alpha,$$

where n, n' denote the indices of refraction of the first and second media, respectively. In the triangles MBC, M'BC, we have:

$$CM : BM = \sin\alpha : \sin\phi, \qquad CM' : BM' = \sin\alpha' : \sin\phi,$$

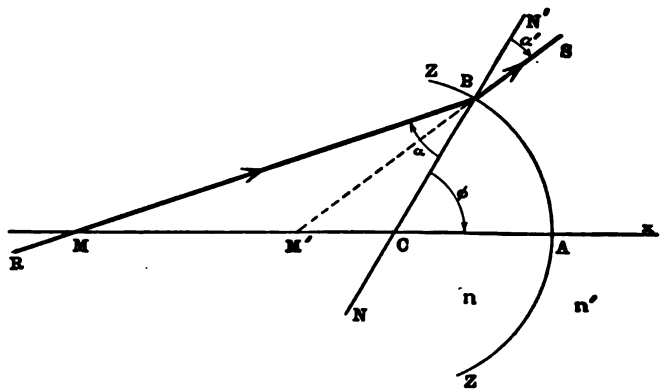


FIG. 104, *b*.—Concave spherical refracting surface ($n' > n$).

where $\phi = \angle BCA$. Dividing one of these equations by the other, we obtain:

$$\frac{CM}{CM'} : \frac{BM}{BM'} = \frac{n'}{n}.$$

Now if the ray RB is a paraxial ray, the incidence-point B will be so near the vertex A of the spherical refracting surface that A may be written in place of B, according to the definition of a paraxial ray as given in § 63. Therefore, in the case of the refraction of paraxial rays at a spherical surface the four points C, A, M, M' on the axis are connected by the following relation:

$$\frac{CM}{CM'} : \frac{AM}{AM'} = \frac{n'}{n}$$

hich may be written (§ 65):

$$(\mathrm{CAMM'}) = \frac{n'}{n};$$

hat is, *the double ratio of the four axial points* C, A, M, M′ *s constant and equal to the relative index of refraction from he first medium to the second.*

Thus, for a given spherical surface (that is, for known positions of the points A and C), separating a pair of media of known relative index of refraction (n'/n), the point M′ on the axis corresponding to a given position of the axial point M has a perfectly definite position, entirely independent of the actual slope of the incident paraxial ray RB; whence it may be inferred that M′ is the image of M, so that *to a homocentric bundle of incident paraxial rays with its vertex lying on the axis of the spherical refracting surface there corresponds also a homocentric bundle of refracted rays with its vertex on the axis.*

In Fig. 104, *a* the image at M′ is real, whereas in Fig. 104, *b* it is virtual. Since the relative index of refraction is never less than zero, the value of the double ratio (CAMM′) in the case of refraction at a spherical surface is necessarily positive; consequently, the pair of conjugate points M, M′ is not "separated" (§ 65) by the pair of points A, C, as was found to be the case in reflection at a spherical mirror (§ 68). Thus, if M, M′ designate the positions of a pair of conjugate axial points with respect to a spherical refracting surface, it is always possible to pass from M to M′ along the axis one way or the other without going through either of the points A or C, although in order to do this it may sometimes be necessary to pass through the infinitely distant point of the axis (see § 65). Accordingly, depending only on the form of the surface and on whether n is greater or less than n', there will be found to be sixteen possible orders of arrangement of these four points, viz.:

A, C, M, M′; A, C, M′, M; A, M, M′, C; A, M′, M, C; M, A, C, M′; M′, A, C, M; M, M′, A, C; M′, M, A, C;

together with the eight other arrangements obtained by reversing the order of the letters in each of these combinations; in other words, exactly the series of combinations that are not possible in the case of a spherical mirror where the pair of conjugate axial points M, M′ is harmonically separated by the pair of points A, C, so that $(CAMM') = -1$ (§ 68). The student should draw a diagram similar to Fig. 104 for each of the possible arrangements of the four points above mentioned. Fig. 104, *a* shows the case M, A, C, M′ and Fig. 104, *b* shows the case M, M′, C, A.

Moreover, if $(CAMM') = n'/n$, then also $(CAM'M) = n/n'$, as follows from the definition of the double ratio (§ 65). Consequently, if a paraxial ray is refracted at a point B of a spherical surface from medium n to medium n' along the broken line RBS, a ray directed from S to B will be refracted from medium n' to medium n in the direction BR; which is in accordance with the general principle of the reversibility of the light-path (§ 29). If therefore M′ is the image of M when the light is refracted across the spherical surface in a given sense, then also M will be the image of M′ when the refraction takes place in the reverse sense.

75. Reflection Considered as a Special Case of Refraction.—It was implied above that if it were possible for the ratio n'/n to have not only positive values but also the unique negative value -1, the single formula $(CAMM') = n'/n$ would express the relation between a pair of conjugate axial points M, M′ both for a spherical refracting surface and for a spherical mirror. The question naturally arises, therefore, Is there a general rule of this kind applicable also to other problems in optics that are not necessarily concerned with paraxial rays or particular conditions? Returning to fundamental principles and recalling the laws of reflection and refraction, we observe that while the angles of incidence and refraction always have like signs, the angles of incidence and reflection, on the contrary, have opposite signs. In order, therefore, that the refraction-formula $n' . \sin\alpha' = n . \sin\alpha$

may include also the law of reflection as well, the values of n and n' in the latter case must be such that $a' = -a$ is a solution of the equation in question; and obviously this solution can be obtained only by putting

$$n' = -n, \text{ or } \frac{n'}{n} = -1.$$

Accordingly, the rule discovered above to be true in a special case is found to be entirely general, so that, at least from a purely mathematical point of view, the reflection of light may be regarded as a particular case of refraction back again into the medium of the incident light, provided we assign to this medium two equal and opposite values of the absolute index of refraction. The convenience of this artifice is apparent, since it makes it quite unnecessary to investigate separately and independently each special problem of reflection and refraction; for when in any given case the relation between an incident ray and the corresponding refracted ray has been ascertained, it will be necessary merely to impose the condition $n' = -n$ in order to derive immediately the analogous relation between the incident ray and the corresponding reflected ray. Thus, for example, any formula hereafter to be derived concerning the refraction of paraxial rays at a spherical surface may be converted into the corresponding formula for the case of a spherical mirror by putting $n' = -n$.

76. Construction of the Point M′ Conjugate to the Axial Point M.—In order to construct the point M′ conjugate to the axial point M with respect to a spherical refracting surface, we may proceed as follows:

Through the vertex A (Fig. 105, *a*, *b*, *c* and *d*) and the center C draw a pair of parallel straight lines (preferably but not necessarily) at right angles to the axis; and on the line going through C take two points O and O′ such that

$$CO : CO' = n' : n.$$

Join the given axial point M by a straight line with the point O, and let B designate the point where this straight line,

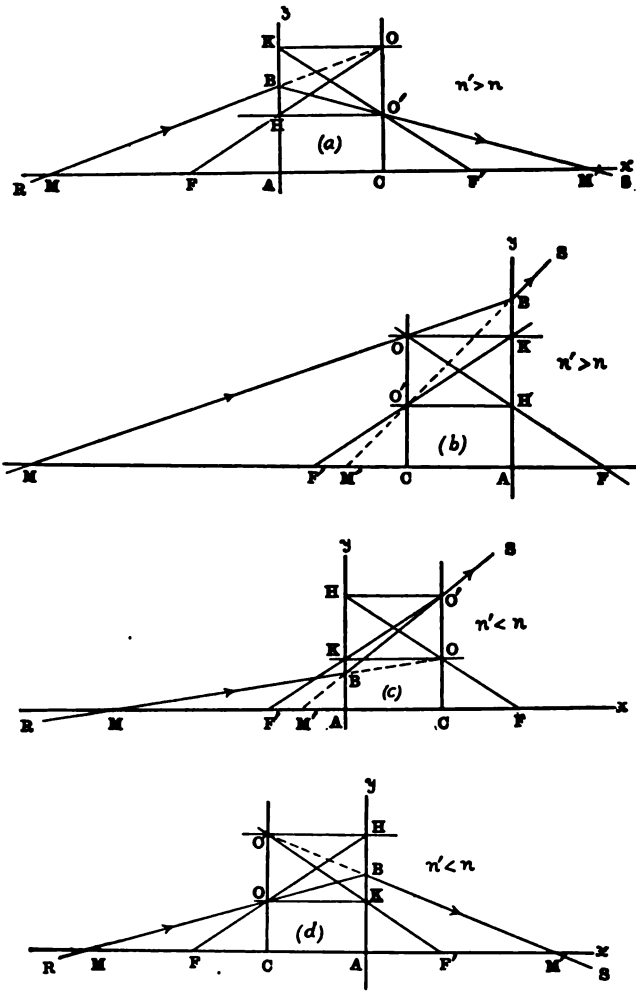


FIG. 105.—Spherical refracting surface: Construction of image-point M′ conjugate to axial object-point M; construction of focal points F, F′.
(*a*) Convex surface, $n' > n$; order MACM′.
(*b*) Concave surface, $n' > n$; order MM′CA.
(*c*) Convex surface, $n' < n$; order MM′AC.
(*d*) Concave surface, $n' < n$; order MCAM′.

ıced if necessary, meets the line drawn through A el to CO; then the required point M′ will be at the where the straight line BO′, produced if necessary, ects the axis.

ɜ straight line A*y* drawn perpendicular to the axis at A e tangent to the spherical surface at its vertex; and this ′ill represent the spherical surface in the diagram, since e concerned here only with paraxial rays (§ 70). Thus, ɜ incident ray RB crossing the axis at M and incident e surface at B, there will correspond the refracted ray ossing the axis at M′.

ɜ proof of the construction consists in showing that ɔuble ratio (CAMM′) is equal to $\frac{n'}{n}$, in accordance with ɨlation which, as we saw above (§ 74), connects the two gate points M, M′.

the pair of similar triangles CMO, AMB,

$$CM : AM = CO : AB;$$

ı the pair of similar triangles CM′O′, AM′B,

$$AM' : CM' = AB : CO'.$$

plying these two proportions, we obtain:

$$\frac{CM}{CM'} \cdot \frac{AM'}{AM} = \frac{CO}{CO'},$$

$$\frac{CM}{CM'} : \frac{AM}{AM'} = \frac{n'}{n},$$

ence

$$(CAMM') = \frac{n'}{n}.$$

e diagrams illustrate four cases, viz., the cases when oints A, C, M, M′ are ranged along the axis from left ght in the orders MACM′, MM′CA, MM′AC and M′. In the diagrams Fig. 105, *a* and *b*, the second ım is represented as more highly refracting than the

first ($n'>n$), whereas in the two other diagrams Fig. 105, *c* and *d*, the opposite case is shown ($n'<n$); in *a* and *c* the surface is convex, and in *b* and *d* it is concave.

77. The Focal Points (F, F′) of a Spherical Refracting Surface.—The object-point F which is conjugate to the infinitely distant image-point E and the image-point F′ which is conjugate to the infinitely distant object-point E of the axis are the so-called *focal points* of the spherical refracting surface. A conical bundle of incident paraxial rays with its vertex at the *primary focal point* F will be converted into a cylindrical bundle of refracted rays all parallel to the axis and meeting therefore in the infinitely distant point E of the axis; and, similarly, a cylindrical bundle of paraxial rays proceeding from the infinitely distant point E of the axis will be transformed into a conical bundle of refracted rays with its vertex at the *secondary focal point* F′.

According to the method explained in § 76, the focal point F may be constructed by drawing the straight line O′H (Fig. 105, *a*, *b*, *c* and *d*) through O′ parallel to the axis meeting the straight line AB in the point designated by H; and then the straight line OH will intersect the axis in the primary focal point F. Similarly, if the straight line OK is drawn through O parallel to the axis meeting AB in a point K, the point of intersection of the straight line KO′ with the axis will determine the position of the secondary focal point F′. In brief, the diagonals of the parallelogram OO′HK meet the axis in the focal points F, F′. The spherical refracting surface is said to be *convergent* or *divergent* according as the focal point F′ is real or virtual, respectively. Thus, in the diagrams Fig. 105, *a* and *d*, incident rays parallel to the axis are brought to a real focus at F′, so that the surface is convergent for each of these cases; whereas in the diagrams Fig. 105, *b* and *c*, incident rays parallel to the axis are refracted as if they proceeded from a virtual focus at F′.

Moreover, certain characteristic metric relations may be derived immediately from the diagrams Fig. 105, *a*, *b*, *c*, and *d*.

For example, in the two pairs of similar triangles FAH, HO′O and F′CO′, O′HK, we obtain the proportions:

$$FA : HO' = AH : O'O, \qquad CF' : HO' = CO' : HK,$$

and since $CO' = AH$, $HK = O'O$, we find:

$$FA = CF';$$

and hence also:

$$F'A = CF.$$

Accordingly, concerning the positions of the focal points of a spherical refracting surface we have the following rule:

The focal points of a spherical refracting surface lie on the axis at such places that the step from one of them to the center is identical with the step from the vertex to the other focal point.

This statement should be verified for each of the diagrams. Not only will the center C be seen to be at the same distance from the primary focal point as the secondary focal point is from the vertex A, but the direction from F to C will always be the same as that from A to F′.

This relation may also be expressed in a different way; for, since

$$FA = CF' = CA + AF',$$

we have the following equation:

$$FA + F'A = CA; \text{ or } AC = AF + AF';$$

which may be put in words by saying that *the step from the vertex to the center of a spherical refracting surface is equal to the sum of the steps from the vertex to the two focal points.*

And, finally, since in the pair of similar triangles FAH, FCO, we have:

$$FC : FA = CO : AH = CO : CO' = n' : n,$$

and since $FC = -CF = -F'A$, we obtain also another useful and important relation, viz.:

$$\frac{F'A}{FA} = -\frac{n'}{n},$$

and, consequently: *The two focal points* F, F′ *of a spherical refracting surface lie on opposite sides of the vertex* A, *and at distances from it which are in the ratio of* n *to* n'. If, therefore, we are given the positions of one of the two focal points,

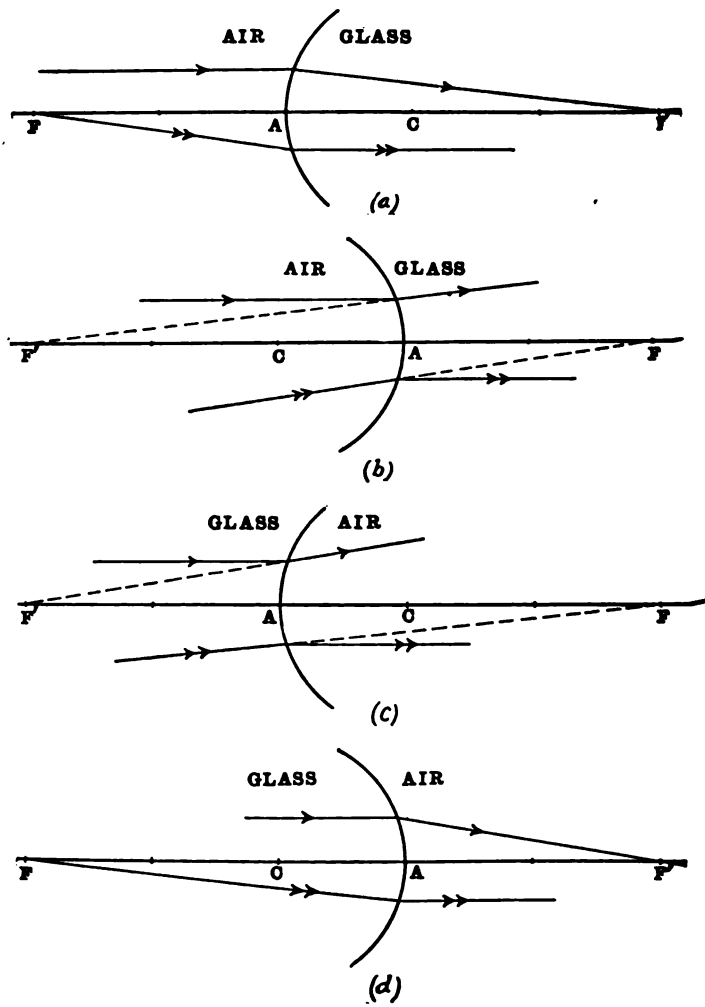


FIG. 106, *a*, *b*, *c* and *d*.—Focal points of spherical refracting surface separating air, of index 1, and glass, of index 1.5.

(*a*) Refraction from air to glass at convex surface.
(*b*) " " " " " " concave "
(*c*) " " glass to air " convex "
(*d*) " " " " " " concave "

F or F′, as well as the positions of the points A, C which determine the size and form of the spherical surface, we have all the data necessary to enable us to locate the point M′ conjugate to a given axial object-point M. For we can locate the position of the other focal point and thus determine the value of the ratio $n' : n$.

Whether the secondary focal point will lie on one side or the other of the spherical refracting surface, that is, whether the surface will be convergent or divergent, will depend on each of two things, viz.: (1) Whether the surface is convex or concave, and (2) Whether n' is greater or less than n. For example, if the rays are refracted from air to glass ($n'/n = 3/2$), according to the above relations we find that AF = 2 CA, AF′ = 3 AC; so that starting at the vertex A and taking the step CA twice we can locate the primary focal point F; and returning to the vertex A and taking the step AC three times, we arrive at the secondary focal point F′. The diagrams Fig. 106, *a* and *b*, show the positions of the focal points for refraction from air to glass for a convex surface and for a concave surface. In this case the convex surface is convergent and the concave surface is divergent. On the other hand, when the light is refracted from glass to air ($n'/n = 2/3$), we find AF = 3 AC, AF′ = 2 CA (Fig. 106, *c* and *d*), and in this case the concave surface is convergent and the convex surface is divergent.

In conclusion, it may be added that the constructions and rules which have been given above for the case of a spherical refracting surface are entirely applicable also to a *spherical mirror*. In fact, here we have an excellent illustration of the method of treating reflection as a special case of refraction, which was explained in § 75. For if we take $n' = -n$, the two points O, O′ (Fig. 107, *a* and *b*) will lie on a straight line passing through the center C of the mirror at equal distances from C in opposite directions. The point M′ conjugate to the axial object-point M and the focal points F, F′ will be found precisely according to the directions for

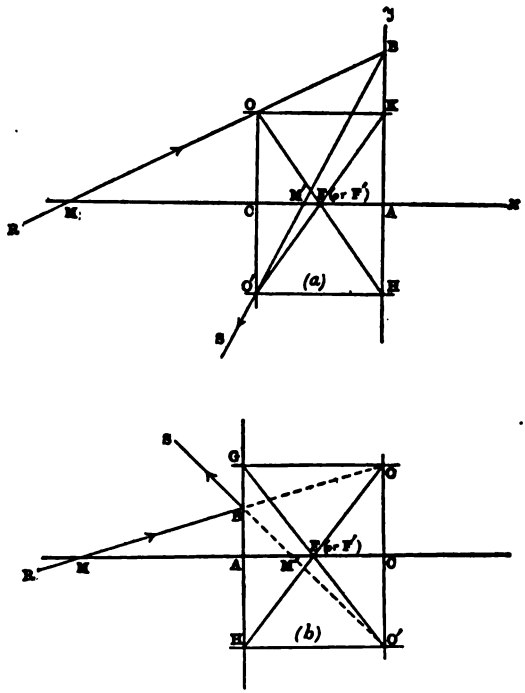


FIG. 107, *a* and *b*.—Reflection at spherical mirror: Construction of image-point M′ conjugate to axial object-point M; construction of focal point.
(*a*) concave mirror, (*b*) convex mirror.

drawing the diagrams of Fig. 105. Obviously, the focal points of a spherical mirror will coincide with each other at a point midway between the vertex and center (§ 69).

78. Abscissa-Equation referred to the Vertex of the Spherical Refracting Surface as Origin.—If the vertex A of the spherical refracting surface is taken as the origin (§ 63) from which distances or steps along the axis are reckoned, and if the symbols r, u and u' are employed as in the case of a spherical mirror (§ 64) to denote the abscissæ of the

center C and the pair of conjugate axial points M, M′, that is, if $AC=r$, $AM=u$, $AM'=u'$, then

$$CM=CA+AM=u-r, \qquad CM'=CA+AM'=u'-r;$$

and since the formula $(CAMM')=\frac{n'}{n}$ may evidently be written as follows:

$$n'.\frac{CM'}{AM'}=n.\frac{CM}{AM},$$

we obtain:

$$n'.\frac{u'-r}{u'}=n.\frac{u-r}{u}.$$

Dividing both sides by r, we derive the so-called *invariant relation* in the case of refraction of paraxial rays at a spherical surface, in the following form:

$$n'\left(\frac{1}{r}-\frac{1}{u'}\right)=n\left(\frac{1}{r}-\frac{1}{u}\right).$$

Usually, however, this equation is written as follows:

$$\frac{n'}{u'}=\frac{n}{u}+\frac{n'-n}{r},$$

which is to be regarded as one of the fundamental formulæ of geometrical optics. If the two constants r and n'/n are known, the abscissa u' corresponding to any given value of u may easily be determined. Putting $n'=-n$ (§ 75), we obtain the abscissa-formula for reflection of paraxial rays at a spherical mirror (§ 64); and if we put $r=\infty$, we derive the formula $\frac{n'}{u'}=\frac{n}{u}$ for the refraction of paraxial rays at a plane surface (§ 41). It is because this linear equation connecting the abscissæ of a pair of conjugate axial points includes these other cases also that some writers have proposed that the formula above should be called the *characteristic equation of paraxial imagery*.

79. The Focal Lengths f, f′ of a Spherical Refracting Surface.—*The steps from the focal points* F *and* F′ *to the vertex*

A *are called the focal lengths of the spherical refracting surface; the primary focal length, denoted by f, is the abscissa of* A *with respect to* F ($f=\mathrm{FA}$), *and the secondary focal length, denoted by f', is the abscissa of* A *with respect to* F' ($f'=\mathrm{F'A}$).

Since $\mathrm{FA}+\mathrm{F'A}=\mathrm{CA}$ (§ 77), and since $\mathrm{CA}=-r$, the focal lengths and the radius of the surface are connected by the following relation:

$$f+f'+r=0,$$

and hence if two of these magnitudes are known, the value of the third may always be determined from the fact that their algebraic sum is equal to zero. For example, starting at any point on the axis and taking in succession in any order the three steps denoted by f, f' and r, one will find himself at the end of the last step back again at the starting point.

Moreover, the focal lengths are connected with the indices of refraction by the following relation (§ 77):

$$\frac{f'}{f}=-\frac{n'}{n} \text{ or } n.f'+n'.f=0;$$

and, hence, *the focal lengths of a spherical refracting surface are opposite in sign and in the same numerical ratio as that of the indices of refraction.* This formula, as we shall see (§ 122), represents a general law of fundamental importance in geometrical optics.

Expressions for the focal lengths in terms of the radius r and the relative index of refraction ($n' : n$) may be derived immediately from the pair of simultaneous equations above by solving them for f and f'. The same expressions may likewise be easily obtained by substituting in succession in the abscissa-formula (§ 78) the two pairs of corresponding values, viz., $u=-f$, $u'=\infty$ and $u=\infty$, $u'=-f'$. And, finally, they may also be obtained geometrically from one of the diagrams of Fig. 107 by observing that, since by construction $\mathrm{CO} : \mathrm{CO'}=n' : n$, it follows that

$$\mathrm{CO'} : \mathrm{O'O}=n : (n'-n), \qquad \mathrm{CO} : \mathrm{O'O}=n' : (n'-n).$$

Now from the two pairs of similar triangles FAH, HO′O and F′AK, O′HK we obtain the two proportions:

$$FA : HO' = AH : O'O, \qquad F'A : O'H = AK : HK;$$

and since

$FA = f$, $HO' = AC = r$, $AH = CO'$, $F'A = f'$, $AK = CO$, and $HK = O'O$,

we have, finally:

$$f = \frac{n}{n'-n} \cdot r, \quad f' = -\frac{n'}{n'-n} \cdot r;$$

which are exceedingly useful forms of the expressions for the focal lengths.

Since

$$\frac{n}{f} = \frac{n'-n}{r} = -\frac{n'}{f'},$$

the abscissa-relation connecting u and u' may be expressed in terms of one of the focal lengths instead of in terms of the radius r, for example, in terms of the focal length f, as follows:

$$\frac{n'}{u'} = \frac{n}{u} + \frac{n}{f}.$$

80. Extra-Axial Conjugate Points; Conjugate Planes of a Spherical Refracting Surface.—If the axis AC of a spherical refracting surface is revolved in a meridian plane through a very small angle about an axis perpendicular to this plane at the center C, so that the vertex of the surface is displaced a little to one side of its former position A to a point U on the surface, the pair of conjugate points M, M′ will likewise undergo slight displacements into the new positions Q, Q′; and, evidently, the same relation will connect the four points C, U, Q, Q′ on the central line UC as exists between the four points C, A, M, M′ on the axis AC, and accordingly (§ 76) we may write:

$$(CUQQ') = \frac{n'}{n};$$

and hence it is obvious that the points Q, Q′ are a pair of extra-axial conjugate points with respect to the spherical refracting surface. Thus, if the points belonging to an object are all congregated in the immediate vicinity of the axis on an element of a spherical surface which is concentric with the refracting sphere, the corresponding image-points will all be assembled on an element of another concentric spherical surface, and any straight line going through C will determine by its intersections with this pair of concentric surfaces two conjugate points Q, Q′. In order that the rays concerned may all be incident near the vertex A, it is necessary to assume that ∠UCA is very small, which means that the little elements of the surfaces described around C may in fact be regarded as plane surfaces perpendicular to the axis AC. Accordingly, the imagery produced by the refraction of paraxial rays at a spherical surface may be described by the following statements:

(1) *The image of a plane object perpendicular to the axis of a spherical refracting surface is similar to the object, and will lie likewise in a plane perpendicular to the axis;* (2) *A straight line drawn through the center* C *will intersect a pair of conjugate planes in a pair of conjugate points* Q, Q′; and (3) *Incident rays which interesct in* Q *will be transformed into refracted rays which intersect in* Q′.

Diagrams showing the refraction of paraxial rays at a spherical surface should be drawn therefore according to the plan explained in § 70, as has been already stated. The spherical refracting surface *must* be represented in the figure by the plane tangent to the surface at its vertex A, whose trace in the meridian plane of the drawing is the straight line Ay which is taken as the y-axis of the system of rectangular coördinates whose origin is at A (§ 63).

81. Construction of the Point Q′ which with Respect to a Spherical Refracting Surface is Conjugate to the Extra-axial Point Q.—The point Q′ conjugate to the extra-axial point Q is easily constructed. Having first located the focal

points F, F′ (§ 77), we draw through Q (Figs. 108 and 109) a straight line parallel to the x-axis meeting the y-axis in the point designated by V; then the point of intersection of the

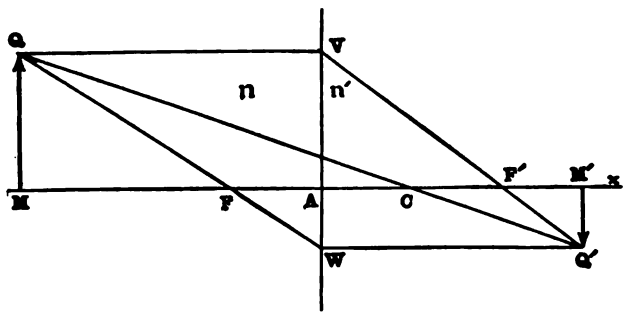


Fig. 108.—Spherical refracting surface: Lateral magnification and construction of image. Convex surface, $n' > n$.

straight lines VF′ and QC will be the required point Q′. A third line may also be drawn through Q, viz., the straight line QF meeting the y-axis in the point marked W; and if a

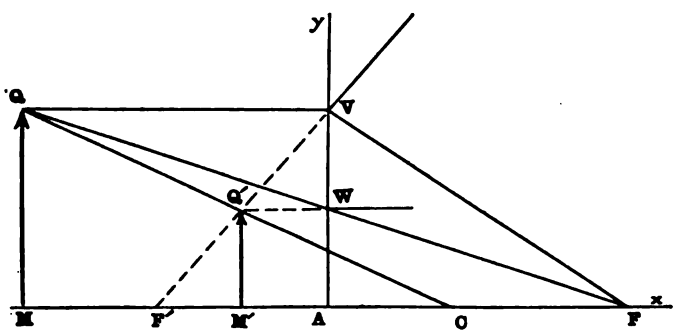


Fig. 109.—Spherical refracting surface: Lateral magnification and construction of image. Concave surface, $n' > n$.

straight line is drawn through W parallel to the x-axis, it will likewise pass through Q′.

If M, M′ designate the feet of the perpendiculars let fall

from Q, Q′ respectively, on the x-axis, then M′Q′ will be the image of the small object-line MQ. In Fig. 108, which represents the case of a convex refracting surface, the image is real and inverted, whereas in Fig. 109 the surface is concave and the image is virtual and erect. Both diagrams are drawn for the case when $n' > n$.

If the object-point Q coincides with the point marked V, the image-point Q′ will also be at V, and image and object will be congruent. The pair of conjugate planes of an optical system for which this is the case are called the *principal planes* (see § 119); and hence *the principal planes of a spherical refracting surface coincide with each other and are identical with the tangent-plane at the vertex.*

82. Lateral Magnification for case of Spherical Refracting Surface.—The ratio M′Q′ : MQ (Figs. 108 and 109) is the so-called *lateral magnification* of the spherical refracting surface with respect to the pair of conjugate axial points M, M′. Since

$$M'Q' : MQ = CM' : CM,$$

and since (§ 74)

$$\frac{CM'}{CM} = \frac{n'}{n} \cdot \frac{AM'}{AM},$$

we find:

$$\frac{M'Q'}{MQ} = \frac{n}{n'} \cdot \frac{AM'}{AM}.$$

If y, y' denote the heights of object and image, that is, if $y = MQ$, $y' = M'Q'$, and if we put the lateral magnification equal to $\boldsymbol{y}$, as in § 72, then, evidently:

$$\boldsymbol{y} = \frac{y'}{y} = \frac{n}{n'} \cdot \frac{u'}{u},$$

where $u = AM$, $u' = AM'$. The lateral magnification depends, therefore, on the position of the object, and the image is erect or inverted according as this ratio is positive or negative.

83. The Focal Planes of a Spherical Refracting Surface.—The focal planes are the pair of planes which are perpendicular to the axis at the focal points F, F'. "The infinitely distant plane of space," which, according to the notions of the modern geometry, is to be regarded as the locus of the infinitely distant points (§ 65) of space, is the image-plane conjugate to the *primary focal plane*, which is the plane perpendicular to the axis at F. On the other hand, regarded as belonging to the object-space, the infinitely distant plane is imaged by the *secondary focal plane* perpendicular to the axis at F'.

The rays proceeding from an infinitely distant object-point I (Fig. 110) constitute a cylindrical bundle of parallel in-

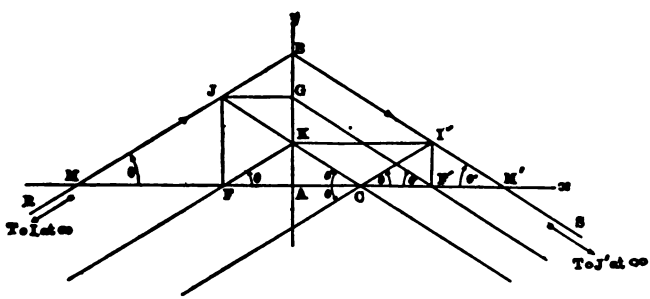


FIG. 110.—Focal planes and focal lengths of spherical refracting surface.

cident rays. Since I lies in the infinitely distant plane of space, its image I' will be formed in the secondary focal plane, and the position of I' in this plane may be located by drawing through the center C of the spherical refracting surface a straight line parallel to the system of parallel rays which meet in the infinitely distant point I. Thus, for example, the image of a star which may be regarded as a point infinitely far away will be formed in the secondary focal plane; and if the apparent place of the star in the firmament is in the direction CI, the star's image will be at the

point I′ where the straight line CI meets the secondary focal plane.

Similarly, if J designates the position of an object-point lying in the primary focal plane, its image J′ will be the infinitely distant point of the straight line JC. Thus, *to a homocentric bundle of incident paraxial rays with its vertex in the primary focal plane, there corresponds a cylindrical bundle of refracted rays; and to a cylindrical bundle of incident paraxial rays there corresponds a homocentric bundle of refracted rays with its vertex in the secondary focal plane.*

The directions of the infinitely distant points I and J′ are given by assigning the values of the slope-angles

$$\theta = \angle \mathrm{FCI} = \angle \mathrm{F'CI'}, \qquad \theta' = \angle \mathrm{FCJ} = \angle \mathrm{F'CJ'};$$

and the points I′ and J conjugate to them will lie in the secondary and primary focal planes on straight lines passing through the center C and inclined to the axis at the angles θ and θ', respectively. The angle θ, which is the measure of the angular distance from the axis of the infinitely distant object-point I, determines the *apparent size* of an object in the infinitely distant plane of the object-space; and, similarly, the angle θ' is the measure of the apparent size of the infinitely distant image of the object FJ.

Draw the straight lines JG and I′K paralled to the optical axis and meeting the y-axis in the points designated by G and K, respectively; then the straight lines FK and CI′ will be parallel to each other, and the same will be true with respect to the straight lines GF′ and JC. Hence, $\angle \mathrm{AFK} = \theta$, $\angle \mathrm{AF'G} = \theta'$; and since $\mathrm{AK} = \mathrm{F'I'}$ and $\mathrm{AG} = \mathrm{FJ}$, we find:

$$\frac{\mathrm{F'I'}}{\mathrm{FA}} = \tan\theta, \quad \frac{\mathrm{FJ}}{\mathrm{F'A}} = \tan\theta'.$$

Putting $\mathrm{FA} = f$ and $\mathrm{F'A} = f'$ (§ 79), we obtain the following expressions for the focal lengths:

$$f = \frac{\mathrm{F'I'}}{\tan\theta}, \quad f' = \frac{\mathrm{FJ}}{\tan\theta'};$$

and since the tangents of the small angles θ, θ' are indistinguishable from the angles themselves (see § 63), we obtain new definitions of the focal lengths, as follows:

The primary focal length is the ratio of the height of the image, in the secondary focal plane, of an infinitely distant object to the apparent size of the object; and the secondary focal length is the ratio of the height of an object in the primary focal plane to the apparent size of the infinitely distant image.

The ratio of the apparent size of the infinitely distant image to the height of an object in the primary focal plane is a measure of the *magnifying power* of the optical system (see § 158), and in this sense we may say that *the magnifying power of a spherical refracting surface is equal to the reciprocal of the secondary focal length.*

84. Construction of Paraxial Ray Refracted at a Spherical Surface.—The refracted ray corresponding to a paraxial ray IB (Fig. 110) incident on a spherical refracting surface at the point B may easily be constructed, for example, in one of the following ways:

(*a*) Through the primary focal point draw the straight line FK parallel to IB meeting the y-axis in the point K; and through K draw a straight line parallel to the x-axis meeting the secondary focal plane in the point I′; the path of the refracted ray will lie along the straight line BI′.

(*b*) Through the center C draw a straight line CI′ parallel to the given incident ray meeting the secondary focal plane in the point I′; the path of the corresponding refracted ray will be along the straight line BI′.

(*c*) Let J designate the point where the given incident ray crosses the primary focal plane, and draw the straight line JG parallel to the x-axis meeting the y-axis in the point designated by G; then the path of the required refracted ray will lie along the straight line BI′ drawn through the incidence-point B parallel to the straight line GF′, where F′ designates the position of the secondary focal point.

(*d*) Finally, the required refracted ray will be along the straight line BI′ drawn parallel to the straight line JC.

85. The Image-Equations in the case of Refraction of Paraxial Rays at a Spherical Surface.—The rectangular coördinates of the image-point Q′ may easily be expressed in terms of the coördinates of the object-point Q. But the forms of these expressions will depend partly on the particular pair of constants (n'/n, r and f, f') which define the surface and partly on the system of axes to which the coördinates are referred. The axis of the spherical surface will always represent the axis of abscissæ (x-axis), and the y-axis will be at right angles to it; but the origin may be taken at any place along the x-axis. If the vertex A is taken as the origin (§ 63), the coördinates of Q, Q′ will be (u, y) and (u', y'); that is, $u=\mathrm{AM}$, $u'=\mathrm{AM}'$, $y=\mathrm{MQ}$, $y'=\mathrm{M}'\mathrm{Q}'$; and since (§§ 78 and 82)

$$\frac{n'}{u'}=\frac{n}{u}+\frac{n'-n}{r}, \quad \frac{y'}{y}=\frac{n\,u'}{n'u},$$

we obtain by solving for u' and y';

$$u'=\frac{n'ru}{(n'-n)\,u+nr}, \quad y'=\frac{nry}{(n'-n)\,u+nr}.$$

In terms of the same coördinates, but with a different pair of constants, viz., f, f', instead of n': n, r, the image-equations may be put also in other forms, as follows:

It will be recalled that in § 79 the abscissa-formula was written:

$$\frac{n'}{u'}=\frac{n}{u}+\frac{n}{f},$$

and since (§ 79) $n'/n=-f'/f$, n and n' may be eliminated and the image-equations will become:

$$\frac{f}{u}+\frac{f'}{u'}+1=0, \quad \frac{y'}{y}=\frac{f}{f+u}=\frac{f'+u'}{f'}=-\frac{fu'}{f'u};$$

which are also frequently employed. These formulæ may also be easily derived from the geometrical relations in Figs. 108 and 109, since we have the proportions:

$$\mathrm{FM}:\mathrm{AM}=\mathrm{VA}:\mathrm{VW}=\mathrm{AF}':\mathrm{AM}'.$$

Instead of a single system of rectangular coördinates, we may have two systems, one for the object-space and the other for the image-space. For example, if the focal points F, F′ are selected as the origins of two such systems, and if the abscissæ of the pair of conjugate axial points M, M′ are denoted by x, x', that is, if $x = \mathrm{FM}$, $x' = \mathrm{F'M'}$, then, since $u = \mathrm{AM} = \mathrm{AF} + \mathrm{FM} = x - f$, $\quad u' = \mathrm{AM'} = \mathrm{AF'} + \mathrm{F'M'} = x' - f'$, the abscissæ, u, u' may be eliminated from the equations above, and the image-equations will be obtained finally in their simplest forms, as follows:

$$\frac{y'}{y} = \frac{f}{x} = \frac{x'}{f'}.$$

These relations may be derived directly from the two pairs of similar triangles FMQ, FAW and F′M′Q′, F′AV in Figs. 108 and 109. The abscissa-relation

$$x.x' = f.f'$$

is the so-called *Newtonian formula* (see § 69). If the x's are plotted as abscissæ and the x''s as ordinates, this equation will represent a rectangular hyperbola.

86. The so-called Smith-Helmholtz Formula.—In Fig. 111 if $\mathrm{M'Q'} = y'$ represents the image in a spherical refracting

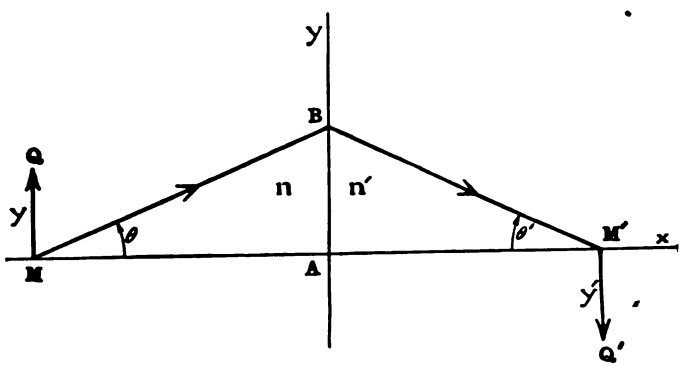


FIG. 111.—Spherical refracting surface: SMITH-HELMHOLTZ law.

surface Ay of a small object-line $\mathrm{MQ} = y$ perpendicular the axis at M, and if B designates the incidence-point o

paraxial ray which crosses the axis before and after refraction at M and M′, respectively, then in the triangle MBM′

$$\sin\theta : \sin\theta' = BM' : BM,$$

where $\theta = \angle AMB$, $\theta' = \angle AM'B$ denote the slopes of the incident ray MB and the corresponding refracted ray BM′. Since the ray is paraxial, we may put $\theta = \sin\theta$, $\theta' = \sin\theta'$ and also $BM = AM = u$, $BM' = AM' = u'$ (§ 63). Hence,

$$\frac{\theta}{\theta'} = \frac{u'}{u}, \text{ or } u' . \theta' = u . \theta.$$

But (§ 82)

$$\frac{n' . y'}{u'} = \frac{n . y}{u};$$

and, therefore, by multiplying these two equations so as to eliminate u and u', we obtain the important invariant-relation in the case of refraction of paraxial rays at a spherical surface, viz.:

$$n' . y' . \theta' = n . y . \theta.$$

This formula states that the function obtained by the continued product of the three factors n, y, θ has the same value after refraction at a spherical surface as it had before refraction. It is a special case of a general law which is found to apply to a centered system of spherical refracting surfaces (§ 118) and which is usually known as LAGRANGE's law; but undoubtedly ROBERT SMITH who announced the law for the case of a system of thin lenses as early as 1738 is entitled to the credit of it. The importance of the relation was recognized by HELMHOLTZ(1821–1894), and the form in which it is written above is due to him. On the whole it seems proper to adopt the suggestion of P. CULMANN and to refer to this equation as the SMITH-HELMHOLTZ formula.

PROBLEMS

1. If A designates the vertex and C the center of a spherical mirror, and if M, M′ designate the points where a paraxial ray crosses the straight line AC before and after reflection, respectively, show that

$$\frac{1}{u}+\frac{1}{u'}=\frac{2}{r},$$

where $r=\mathrm{AC}$, $u=\mathrm{AM}$, $u'=\mathrm{AM}'$.

2. The radius of a concave mirror is 30 cm. Paraxial rays proceed from a point 60 cm. in front of it; find where they are focused after reflection.

Ans. At a point 20 cm. in front of the mirror.

3. The radius of a concave mirror is 60 cm. A luminous point is placed in front of the mirror at a distance of (*a*) 120 cm., (*b*) 60 cm., (*c*) 30 cm., and (*d*) 20 cm. Find the position of the image-point for each of these positions of the object.

Ans. (*a*) 40 cm. in front of mirror; (*b*) 60 cm. in front of mirror; (*c*) at infinity; and (*d*) 60 cm. behind mirror.

4. A candle is placed in front of a concave spherical mirror, whose radius is 1 foot, at a distance of 5 inches from the mirror. Where will the image be formed?

Ans. 30 inches behind the mirror.

5. An object is 24 inches in front of a concave mirror of radius 1 foot; where will its image be formed? If the object is displaced through a small distance z, through what distance will the image move?

Ans. Image is 8 inches in front of mirror; distance through which image moves will be $2z/(z-18)$.

6. An object is placed 1 foot from a concave mirror of radius 4 feet. If the object is moved 1 inch nearer the mirror, what will be the corresponding displacement of the image?

Ans. The image moves 3.7 inches nearer the mirror.

7. An object-point is 10 cm. in front of a convex mirror of radius 60 cm. Find the position of the image-point.

Ans. 7.5 cm. behind the mirror.

8. Given the positions on the axis of a spherical mirror of the vertex A, the center C and an object-point M; show how to construct the position of the image-point M'. There are eight possible arrangements of these four points; draw a diagram for each one of them.

9. If x, x' denote the abscissæ, with respect to the focal point F as origin, of a pair of conjugate points on the axis of a spherical mirror, show that

$$x.x' = f^2,$$

where f denotes the focal length of the mirror. How are object and image situated with respect to the focal plane?

10. An object is placed at a distance of 60 cm. in front of a spherical mirror, and the image is found to be on the same side of the mirror at a distance of 20 cm. What is the focal length of the mirror, and is it concave or convex?

Ans. Concave mirror of focal length 15 cm.

11. How far from a concave mirror of focal length 18 inches must an object be placed in order that the image shall be magnified three times?

Ans. 1 ft. or 2 ft. from the mirror, according as image is erect or inverted.

12. A candle-flame one inch high is 18 inches in front of a concave mirror of focal length 15 inches. Find the position and size of the image.

Ans. The image will be real and inverted, 90 inches from the mirror, and 5 inches long.

13. A small object is placed at right angles to the axis of a spherical mirror; show how to construct the image, and derive the magnification-formula:

$$\frac{y'}{y} = -\frac{u'}{u}.$$

14. A luminous point moves from left to right along a straight line parallel to the axis of a spherical mirror. Show by diagrams for both concave and convex mirrors how the conjugate image-point moves.

15. The center of a spherical mirror is at C, and the

straight line QQ′ joining a pair of conjugate points meets the mirror in a point U. If P designates the position of a point which is not on the straight line QQ′, and if a straight line is drawn cutting the straight lines PU, PQ, PC and PQ′ in the points V, R, Z and R′, respectively; show that R, R′ are a pair of conjugate points with respect to another spherical mirror whose center is at Z and whose radius is equal to VZ.

16. Show by geometrical construction that the focal point of a spherical mirror lies midway between the center and the vertex.

17. An object is placed 5 inches from a spherical mirror of focal length 6 inches. Assuming that the object is real, where will the image be formed, and what will be the magnification? Draw diagrams for both convex and concave mirrors.

Ans. For concave mirror, image is 30 in. behind the mirror, magnification $=+6$; for convex mirror, image is $2\frac{8}{11}$ inches behind the mirror, magnification $=+\frac{6}{11}$.

18. How far from a concave mirror must a real object be placed in order that the image shall be (*a*) real and four times the size of the object, (*b*) virtual and four times the size of the object, and (*c*) real and one-fourth the size of the object? Draw diagrams showing the construction for each of these three cases.

Ans. Distance of mirror from the object is equal to (*a*) $5f/4$, (*b*) $3f/4$, and (*c*) $5f$, where f denotes the focal length.

19. What kind of image is produced in a concave mirror by a virtual object? Illustrate and explain by means of a diagram.

Ans. Image is real and erect and smaller than object.

20. Determine the position and magnification of the image of a virtual object lying midway between the vertex and focal point of a convex mirror. Draw diagram showing construction.

Ans. The vertex of the mirror will be midway between the

axial point of the image and the focal point of the mirror, and the image will be real and erect and twice as large as object.

21. Show that when an object is placed midway between the focal point and the vertex of a concave mirror the image will be virtual and erect and twice as large as the object.

22. An object 3 inches high is placed 10 inches in front of a convex mirror of 30 inches focal length. Find the position and size of the image.

Ans. Virtual image 7.5 inches from the mirror and 2¼ inches high.

23. An object is placed in front of a concave mirror at a distance of one foot. If the image is real and three times as large as the object, what is the focal length of the mirror?

Ans. 9 inches.

24. The radius of a concave mirror is 23 cm. An object, 2 cm. high, is placed in front of the mirror at a distance of one meter. Find the position and size of the image.

Ans. A real image, 0.26 cm. high, 13 cm. from the mirror.

25. Find the position and size of the image of a disk 3 inches in diameter placed at right angles to the axis of a spherical mirror of radius 6 feet, when the distance from the object to the mirror is (*a*) 1 ft., (*b*) 3 ft., and (*c*) 9 ft.

Ans. For a concave mirror: (*a*) Virtual image, 4.5 inches in diameter, 18 inches from mirror; (*b*) Image at infinity; (*c*) Real inverted image, 1.5 inches in diameter, 4.5 feet from the mirror.

26. Assuming that the apparent diameter of the sun is 30′, calculate the approximate diameter of the sun's image in a concave mirror of focal length 1 foot.

Ans. A little more than one-tenth of an inch.

27. A gas-flame is 8 ft. from a wall, and it is required to throw on the wall a real image of the flame which shall be magnified three times. Determine the position and focal length of a concave mirror which would give the required image.

Ans. The mirror must have a focal length of 3 ft. and must be placed at a distance of 4 ft. from the object.

28. It is desired to throw on a wall an image of an object magnified 12 times, the distance of the object from the wall being 11 feet. Find the focal length of a concave mirror which will do this, and state where it must be placed.

Ans. The focal length of the mirror must be $\frac{12}{13}$ ft., and it must be placed 1 ft. from the object.

29. Assuming that the eye is placed on the axis of a spherical mirror, and that the rays are paraxial, explain how the field of view is determined. Draw accurate diagrams for concave and convex mirrors.

30. A man holds, halfway between his eye and a convex mirror 3 feet from his eye, two fine parallel wires, so that they are seen directly and also by reflection in the mirror. Show that if the apparent distance between the wires as seen directly is 5 times that as seen by reflection, the radius of the mirror is 3 feet.

31. A scale etched on a thin sheet of transparent glass is placed between the eye of an observer and a convex mirror of focal length one foot. When the distance between the eye and the scale is three feet, one of the scale divisions appears to cover three divisions of the image in the mirror. Find the position of the mirror.

Ans. The mirror is one foot from the scale.

32. A scale etched on a thin sheet of transparent glass is interposed between the eye of an observer and a convex mirror of focal length f. When the distance of the scale from the eye is b feet, one division of the scale appears to cover m divisions of its image in the mirror. If now the scale is displaced through a distance c in the direction of the axis of the mirror, it is found that one division of the scale appears to cover k divisions in the mirror. Find an expression for f in terms of m, k, b and c.

Ans.

$$f = -\frac{(k-m)\,(b-c)\,bc}{\{b(k-m)-(k-1)c\}\,\{b(k-m)-(k+1)c\}}$$

33. A concave and a convex mirror, each of radius 20 cm., are placed opposite to each other and 40 cm. apart on the same axis. An object 3 cm. high is placed midway between them. Find the position and size of the image formed by reflection, first, at the convex, and then at the concave mirror. Draw accurate diagram, and trace the path of a ray from a point in the object to the corresponding point in the image.

Ans. The image is $12\frac{8}{11}$ cm. from the concave mirror, real and inverted, and $\frac{8}{11}$ cm. high.

34. Same problem as No. 33, except that in this case the image is formed by rays which have been reflected first from the concave mirror and then from the convex mirror.

Ans. The image is $6\frac{2}{3}$ cm. behind the convex mirror, virtual and inverted, and 1 cm. high.

35. Two concave mirrors, of focal lengths 20 and 40 cm., are turned towards each other, the distance between their vertices being one meter. An object 1 cm. high is placed between the mirrors at a distance of 10 cm. from the mirror whose focal length is 20 cm. Find the position and size of the image produced by rays which are reflected first from the nearer mirror and then from the farther mirror.

Ans. A real inverted image, 1 cm. long, at a distance of 60 cm. from the mirror that is farther from the object.

36. The distance between the vertices A_1 and A_2 of two spherical mirrors which face each other is denoted by d, that is, $d = A_2A_1$. The focal points of the mirrors are at F_1 and F_2, and the focal lengths are $f_1 = F_1A_1$ and $f_2 = F_2A_2$. An object is placed between the mirrors at a distance u_1 from A_1. Rays proceeding from the object are reflected, first, from the mirror A_1 and then from the mirror A_2; show that the distance of the final image from the mirror A_2 is

$$\frac{\{f_1.u_1-(f_1+u_1)\ d\}\ f_2}{(f_1+u_1)\ (f_2+d)-f_1.u_1},$$

and that the magnification is

$$\frac{f_1.f_2}{(f_1+u_1)\ (f_2+d)-f_1.u_1}.$$

37. If the rays fall first on the mirror A_2 and then on A_1, these letters having exactly the same meanings as in No. 36, then the distance of the image from mirror A_1 will be

$$\frac{f_1\{(f_2+d)\ (u_1+d)+f_2 d\}}{(f_1-d)\ f_2+(u_1+d)\ (f_1-f_2-d)};$$

and the magnification will be

$$\frac{f_1 . f_2}{(f_1-d)\ f_2+(u_1+d)\ (f_1-f_2-d)}.$$

38. If the mirror A_1 in Nos. 36 and 37 is a plane mirror, show that when the light is reflected from the plane mirror first the distance of the image from the curved mirror is

$$\frac{(u_1-d)\, f_2}{f_2+d-u_1},$$

and that the magnification is

$$\frac{f_2}{f_2+d-u_1};$$

and that when the light is reflected from the curved mirror first, the distance of the image from the plane mirror is

$$\frac{(f_2+d)\ (u_1+d)+f_2 d}{f_2+u_1+d},$$

and that the magnification is

$$\frac{f_2}{f_2+u_1+d}.$$

If both the mirrors are plane, the magnification will be unity, and the image after two reflections, first at A_1 and then at A_2, will be formed at a distance of (u_1-d) from A_2; whereas if the light falls first on mirror A_2, the distance of the image from the other mirror will be (u_1+2d).

39. If M, M′ are a pair of conjugate points on the axis of a spherical refracting surface which divides two media of indices n and n', show that

$$(\mathrm{CAMM'}) = \frac{n'}{n},$$

where A and C designate the vertex and the center of the spherical surface.

40. Show how to construct the position of the point M′

conjugate to a given point M on the axis of a spherical refracting surface; and draw diagrams for all the possible arrangements of the four points A, C, M, M'. Prove the construction, and derive the formula $n'/u' = n/u + (n'-n)/r$, where n, n' denote the indices of refraction, and $u = AM$, $u' = AM'$, $r = AC$.

41. Show how the formula in No. 40 includes as special cases the case of refraction of paraxial rays at a plane surface and the case of reflection at a spherical mirror.

42. From the formula in No. 40 derive expressions for the focal lengths f, f' of a spherical refracting surface, and show that

$$f + f' + r = 0, \quad n.f' + n'.f = 0.$$

43. Does the construction found in No. 40 apply to the case of a spherical mirror? Explain with diagrams.

44. Apply the construction employed in No. 40 to determine the positions of the focal points F, F' of a spherical refracting surface, and show that

$$FA = CF', \quad F'A = CF, \quad F'A : FA = -n' : n.$$

45. Where are the focal points of a plane refracting surface? Explain clearly.

46. Explain how the results of No. 44 are applicable to a spherical mirror.

47. Air and glass are separated by a spherical refracting surface of radius $r = AC$. Find the positions of the focal points F, F' for the cases when the refraction is from air to glass and from glass to air and when the surface is convex and concave; illustrating your answers by four accurately drawn diagrams. (Take indices of refraction of air and glass equal to 1 and 1.5, respectively.)

48. From the figures used in No. 44 for constructing the positions of the focal points F, F', derive the formulæ for the focal lengths which were obtained in No. 42.

49. Light falling on a concave surface separating water ($n = 1.33$) from glass ($n' = 1.55$) is convergent towards a point 10 cm. beyond the vertex. The radius of the surface

is 20 cm. Find the point where the refracted rays cross the axis.

Ans. 13.19 cm. beyond the vertex of the sphere in the glass medium.

50. Light is refracted from air to glass (n': n=3: 2) at a spherical surface. If the vertex of the bundle of incident rays is in the glass and 20 cm. from the vertex of the refracting surface, and if the refracted rays are converged to a point in the glass and 5 cm. from the vertex, determine the form and size of the surface.

Ans. Convex surface of radius 2 cm.

51. A small air-bubble in a glass sphere, 4 inches in diameter, viewed so that the speck and the center of the sphere are in line with the eye, appears to be one inch from the point of the surface nearest the eye. What is its actual distance, assuming that the index of refraction of glass is 1.5?

Ans. 1.2 inches.

52. The radius of a concave refracting surface is 20 cm. A virtual image of a real object is formed at a distance of 40 cm. from the vertex, and the distance from the object to the image is 60 cm. The first medium is air (n=1). Find the index of refraction of the second medium.

Ans. n'=1.6.

53. Light diverging from a point M in air is converged by a spherical refracting surface to a point M′ in glass of index 1.5. The distance MM′=18 cm., and the point M is twice as far from the surface as the point M′. Find the radius of the surface. Ans. 1.5 cm.

54. Find the positions of the focal points F, F′ of a concave spherical refracting surface separating air from a medium of index 1.6, having found that the image of a luminous point 30 cm. in front of the surface is midway between the luminous point and the surface.

Ans. AF=+13.63 cm.; AF′=−21.81 cm.

55. A convergent bundle of rays is incident on a spherical refracting surface of radius 10 cm. The relative index of

refraction from the first medium to the second medium is equal to 2 (n': n=2:1). If the incident rays cross the axis at M and the refracted rays at M′, and if M′M=+60 cm., determine the positions of the points M, M′.

Ans. If the surface is convex, AM=+77.72 cm., AM′ =+17.72 cm. If the surface is concave, then either AM= +30 cm., AM′=−30 cm. or AM=+20 cm., AM′=−40 cm.

56. A beam of parallel rays passing through water (n= 1.3) is refracted at a concave surface into glass (n'=1.5). If the radius of the surface is 20 cm., where will the light be focused? Ans. Virtual focus, 150 cm. from the surface.

57. A small air-bubble is imbedded in a glass sphere at a distance of 5.98 cm. from the nearest point of the surface. What will be the apparent depth of the bubble, viewed from this side of the sphere, if the radius of the sphere is 7.03 cm., and the index of refraction from air to glass is 1.42?

Ans. 5.63 cm.

58. Assuming that the cornea of the eye is a spherical refracting surface of radius 8 mm. separating the outside air from the aqueous humor (of index $\frac{4}{3}$), find the distance of the pupil of the eye from the vertex of the cornea, if its apparent distance is found to be 3.04 mm. Also, if the apparent diameter of the pupil is 4.5 mm., what is its real diameter? Ans. 3.6 mm.; 4 mm.

59. Construct the image M′Q′ of a small object MQ perpendicular at M to the axis of a spherical refracting surface, and derive the magnification-formula in terms of the distances of M and M′ from the vertex of the surface. Draw two diagrams, one for convex, and one for concave surface.

60. Derive the image-equations of a spherical refracting surface referred to the focal points as origins.

61. Derive the image equations of a spherical refracting surface in the forms

$$f/u+f'/u'+1=0, \qquad y'/y=f/(f+u)=(f'+u')/f'.$$

62. Show that there are two positions on the axis of a spherical refracting surface where image and object coincide.

63. Locate the two pairs of conjugate planes of a spherical refracting surface for which image and object have the same size.

64. A real object, 1 cm. high, is placed 12 cm. from a convex spherical refracting surface, of radius 30 cm., which separates air ($n=1$) from glass ($n'=1.5$). Find the position and size of the image.

Ans. Image is virtual and erect, 1.25 cm. high, 22.5 cm. from vertex.

65. In the preceding example, suppose that the object is a virtual object at the same distance from the spherical refracting surface. Find the position and size of the image in this case.

Ans. Image is real and erect, $\frac{5}{8}$ cm. high, and 15 cm. from vertex.

66. Solve Nos. 64 and 65 for the case when the surface is concave; and draw diagrams showing construction of the image in all four cases.

67. Solve No. 64 on the supposition that the first medium is glass and the second medium air.

Ans. Image will be virtual and erect, $1\frac{1}{4}$ cm. high, and $7\frac{1}{2}$ cm. from vertex.

68. (*a*) The human eye from which the crystalline lens has been removed (so-called "aphakic eye") may be regarded as consisting of a single spherical refracting surface, namely, the anterior surface of the cornea. If the radius of this surface is taken as 8 mm., and if the index of refraction of the eye-medium (both the aqueous and vitreous humors) is put equal to $\frac{4}{3}$, what will be the focal lengths of the aphakic eye? (*b*) Assuming that the length of the eye-ball of an aphakic eye is 22 mm., where will an object have to be placed to be imaged distinctly on the retina at the back of the eye?

Ans. (*a*) $f=+24$ mm., $f'=-32$ mm.; (*b*) $u=+52.8$ mm., which means that the object must be virtual and lie behind the eye.

69. LISTING'S "reduced eye" is composed of a single convex spherical refracting surface of radius 5.2 mm. separating air ($n=1$) from the vitreous humor ($n'=1.332$). Calculate the focal lengths.

Ans. $f=+15.68$ mm., $f'=-20.90$ mm.

70. In DONDER'S "reduced eye" the focal lengths are assumed to be $+15$ and -20 mm. Calculate the radius of the equivalent spherical refracting surface and the index of refraction of the vitreous humor for these values of the focal lengths. Ans. $r=+5$ mm.; $n'=\frac{4}{3}$.

71. The angular distance of a star from the axis of a spherical refracting surface which separates air ($n=1$) from glass ($n'=1.5$) is 10°. The surface is convex and of radius 10 cm. Find the position of the star's image.

Ans. A real image will be formed in the secondary focal plane about 3.5 cm. from the axis.

72. What is the size of the image on the retina of LISTING'S "reduced eye" (No. 69) if the apparent size of the distant object is 5°? Ans. 1.36 mm.

73. A hemispherical lens, the curved surface of which has a radius of 3 inches, is made of glass of index 1.5. Show that rays of light proceeding from a point on its axis 4 inches in front of its plane surface will emerge parallel to the axis.

74. A paraxial ray parallel to the axis of a solid refracting sphere of index n' is refracted into the sphere at first towards a point X on the axis, and after the second refraction crosses the axis at a point F'. If the first and last media are the same and of index n, show that the point F' lies midway between the second vertex of the sphere and the point X.

75. A small object of height y is placed at the center of a spherical refracting surface in a plane at right angles to the axis. Determine the position and size of the image. Show how the SMITH-HELMHOLTZ formula (§ 86) is applicable to a part of this problem.

Ans. Image is in same plane as object, erect, and of size $y'=n.y/n'$.

76. A plane object is placed parallel to a plane refracting surface. Show that its image formed by paraxial rays is erect and of same size as object. Is the SMITH-HELMHOLTZ formula (§ 86) applicable to a plane refracting surface? Is it applicable to a spherical mirror? Explain clearly.

77. In a convex spherical refracting surface of radius 0.75, which separates air ($n=1$) from water ($n'=\frac{4}{3}$), the image is real, inverted and one-third the size of the object. Find the positions of object and image. If a ray proceeding from the axial point of the object is inclined to the axis at an angle of 3°, what will be the slope of the corresponding refracted ray?

Ans. Object is in air and image is in water, their distances from the surface being 9 and 4, respectively; slope of refracted ray is $-4.5°$.

78. In a spherical refracting surface

$$\alpha = \theta + \varphi, \quad \alpha' = \theta' + \varphi,$$

where α, α' denote the angles of incidence and refraction, θ, θ' denote the inclinations of the ray to the axis before and after refraction, and φ denotes the so-called central angle ($\angle$BCA). For a paraxial ray the law of refraction may be written

$$n'.\alpha' = n.\alpha.$$

From these formulæ deduce the abscissa-relation in the form

$$\frac{n'}{u'} = \frac{n}{u} + \frac{n'-n}{r}.$$

79. The curved surface of a glass hemisphere is silvered. Rays coming from a luminous point at a distance u from the plane surface are refracted into the glass, reflected from the concave spherical surface, and refracted at the plane surface back into the air. If r denotes the radius of the spherical surface and n the index of refraction of the glass, show that

$$\frac{1}{u} + \frac{1}{u'} + \frac{2n}{r} = 0,$$

where u' denotes the distance of the image from the plane surface.

80. A plane object of height one inch is placed at right angles to the axis of a spherical mirror. The slope of the reflected ray corresponding to an incident paraxial ray which emanates from the axial point of the object at a slope of $+5°$ is $+10°$. Is the image erect or inverted, and what is its size?

Ans. Inverted image, one-half inch high.

CHAPTER VII

REFRACTION OF PARAXIAL RAYS THROUGH AN INFINITELY THIN LENS

87. Forms of Lenses.—In optics the word *lens* is used to denote a portion of a transparent substance, usually isotropic, comprised between two smooth polished surfaces, one of which may be plane. These surfaces are called the faces of the lens. The curved faces are generally spherical, and this may always be considered as implied unless the contrary is expressly stated. Lenses with spherical faces are sometimes called "spherical lenses" to distinguish them from cylindrical, sphero-cylindrical and other forms of lenses which are also quite common, especially in modern spectacle glasses. A plane face may be regarded as a spherical or cylindrical surface of infinite radius.

The *axis* of a lens is the straight line which is normal to both faces, and, consequently, a ray whose path lies along the axis (the so-called axial ray) will pass through the lens without being deflected from this line. The axis of a spherical lens is the straight line joining the centers C_1, C_2 of the two spherical faces, and since a lens of this kind is symmetric around the axis, it may be represented in a plane figure by a meridian section showing the arcs of the two great circles in which this plane intersects the spherical faces. Depending on the lengths of the radii in comparison with the length of the line-segment C_1C_2, these arcs intersect in two points equidistant from the axis or else they do not intersect each other at all.

(*a*) If they intersect, then C_1C_2 is less than the arithmetical sum but greater than the arithmetical difference of the radii, and the lens may be a *double convex lens*

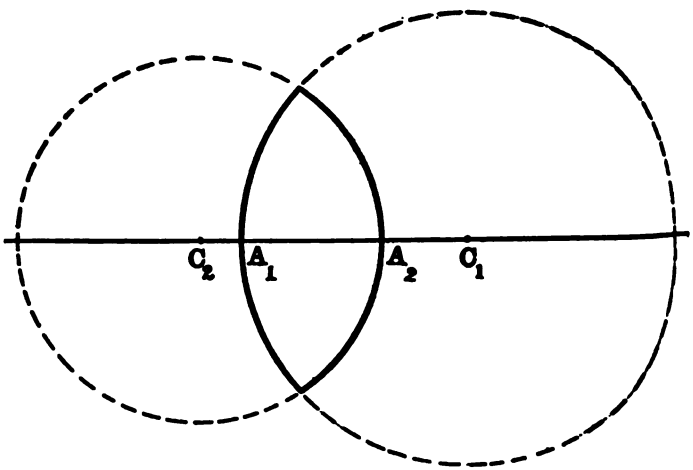


FIG. 112, *a*.—Double convex lens.

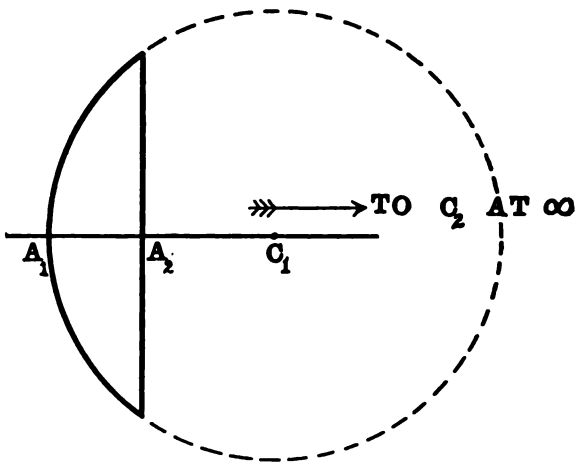


FIG. 112, *b*.—Plano-convex lens.

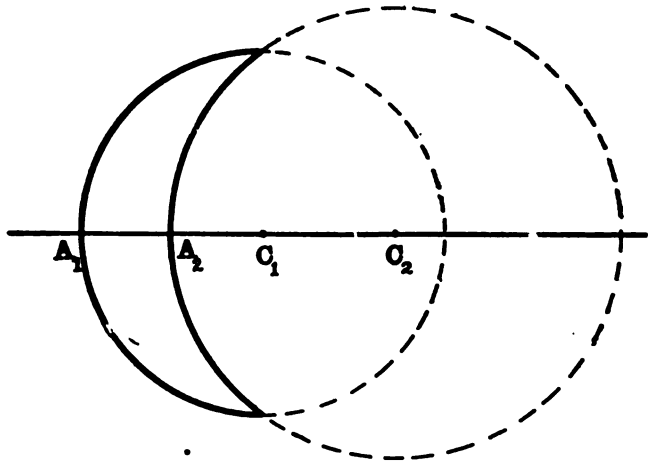


FIG. 112, *c*.—Convex meniscus.

(Fig. 112, *a*) or a *convex meniscus* (Fig. 112, *c*). A particular case of a double convex lens is a *plano-convex lens* (Fig. 112, *b*).

(*b*) If they do not intersect, then either one circle lies wholly outside the other, the distance between the centers being, therefore, greater than the arithmetical sum of the radii, so that the lens is a *double concave lens* (Fig. 113, *a*), or, in case one of the surfaces is plane, a *plano-concave lens* (Fig. 113, *b*); or else one circle lies wholly inside the other, so that the distance between the centers is less than the arithmetical difference of the radii, and then the lens has the form of a *concave meniscus* (Fig. 113, *c*).

The *first face* of a lens is the side turned towards the incident light. The points where the axis meets the two faces are called the *vertices*, and the distance from the vertex A_1 of the first face to the vertex A_2 of the second face, which is denoted by d, is called the thickness of the lens; thus, $d = A_1A_2$. Since the direction which the light takes in going across the lens from A_1 to A_2 is *the positive direction along the axis* (see § 63), the thickness d is essentially a positive magnitude.

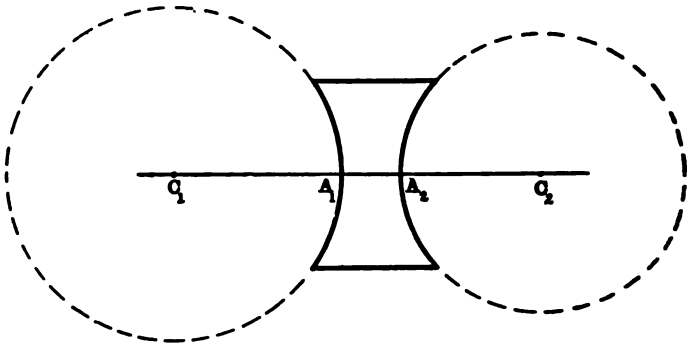


FIG. 113, *a*.—Double concave lens.

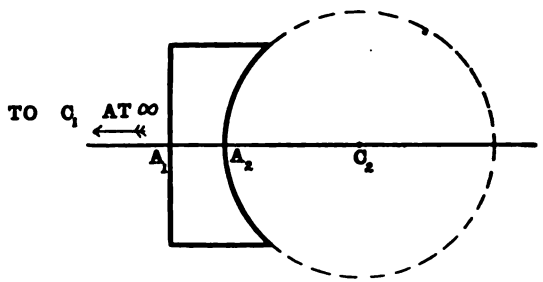


FIG. 113, *b*.—Plano-concave lens.

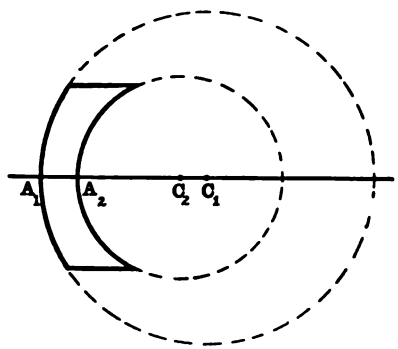


FIG. 113, *c*.—Concave meniscus.

The radii of the surfaces, denoted by r_1, r_2, are the abscissæ of the centers C_1, C_2 with respect to the vertices A_1, A_2, respectively; thus, $r_1 = A_1C_1$, $r_2 = A_2C_2$.

Certain *special forms of spherical lenses* may be mentioned here, viz.:

(*a*) *Symmetric Lenses*, which are double convex or double concave lenses whose surfaces have equal but opposite curvatures ($r_1 + r_2 = 0$). A particular case of double convex symmetric lens is one whose two faces are portions of the same spherical surface; a lens of this kind being sometimes called a *solid sphere* ($d = r_1 - r_2 = 2r_1$).

(*b*) *Concentric Lenses*, whose two faces have the same center of curvature ($C_1C_2 = 0$). A concentric lens may be

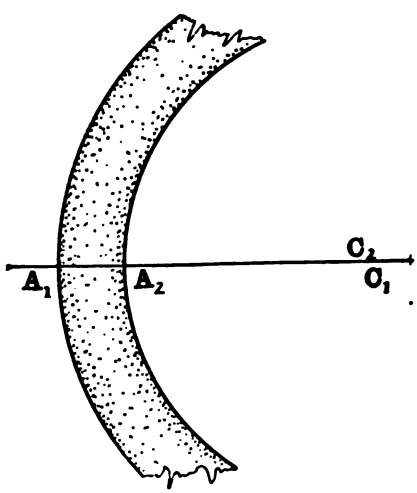


FIG. 114.—Concentric concave meniscus.

a double convex lens characterized by the relation $d = r_1 - r_2$, of which a "solid sphere" is a special case; or it may have the form of a concave meniscus for which either $r_1 > r_2 > 0$ and $d = r_1 - r_2$ (Fig. 114) or $r_1 < r_2 < 0$ and $d = r_2 - r_1$.

(*c*) *Lenses of Zero Curvature*, in which the axial thickness

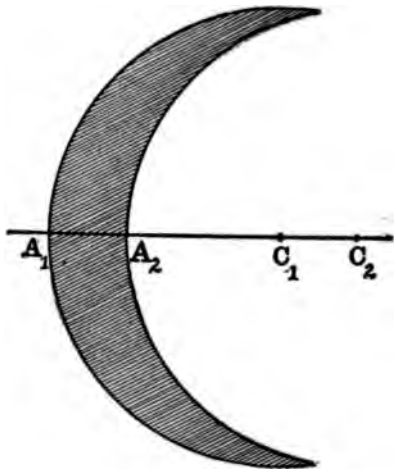


FIG. 115.—Lens of zero curvature ($r_1 = r_2$).

of the lens is equal to the distance between the centers ($d = A_1A_2 = C_1C_2$). This lens is a convex meniscus characterized by the condition that $r_1 - r_2 = 0$ (Fig. 115).

Lenses may also be conveniently classified in two main groups, viz., *convex lenses* and *concave lenses*, depending on the relative thickness of the lens along the axis as compared with its thickness at the edges. The thickness of a convex lens is greater along the axis than it is out towards the edge, whereas a concave lens is thinnest in the middle. Each of these two main divisions includes three special forms which have already been mentioned. Thus, the three types of convex lenses are the double convex, the plano-convex and the convex or "crescent-shaped" meniscus, as shown in Fig. 112; and, similarly, the types of concave lenses are the double concave, plano-concave and the concave or "canoe-shaped" meniscus (Fig. 113).

A convex glass lens of moderate thickness held in air with its axis towards the sun has the property of a burning glass and converges the rays to a real focus on the other side of

the lens. A convex lens is called therefore also a *convergent lens* or a *positive lens*. On the other hand, under the same circumstances, a concave lens will render a beam of sunlight divergent, and, accordingly, a concave lens is called also a *divergent lens* or a *negative lens*. The explanation of the terms "positive" and "negative" as applied to lenses will be apparent when we come to speak of the positions of the focal points of a lens (§ 90).

Finally, if the curvatures of the two faces of the lens are opposite in sign, the lens is double convex or double concave; if the curvatures have the same sign, the lens is a meniscus; and if the curvature of one face is zero, the lens is plano-convex or plano-concave.

88. The Optical Center O of a Lens surrounded by the same medium on both sides.—When a ray of light emerges at the second face of a lens into the surrounding medium in the same direction as it had when it met the first face, the path of the ray inside the lens lies along a straight line which crosses the axis at a remarkable point O called the *optical center* of the lens, which is indeed the (internal or external) "center of similitude" of the two circles whose arcs are the traces of the spherical faces of the lens in the meridian plane which contains the ray.

In order to prove this, draw a pair of parallel radii C_1B_1 and C_2B_2 (Fig. 116), and suppose that a ray enters the lens at B_1 and leaves it at B_2, so that the straight line B_1B_2 represents the path of the ray through the lens. If the straight line RB_1 represents the path of the incident ray, a straight line B_2S drawn through B_2 parallel to RB_1 will represent the path of the emergent ray; because, since the tangents to the circular arcs at B_1, B_2 are parallel to each other, the lens behaves towards this ray which enters it at B_1 exactly like a slab of the same material with plane parallel sides (§ 44). Consequently, the position of the point O where the straight line B_1B_2, produced if necessary, crosses the axis of the lens is seen to be entirely dependent on the

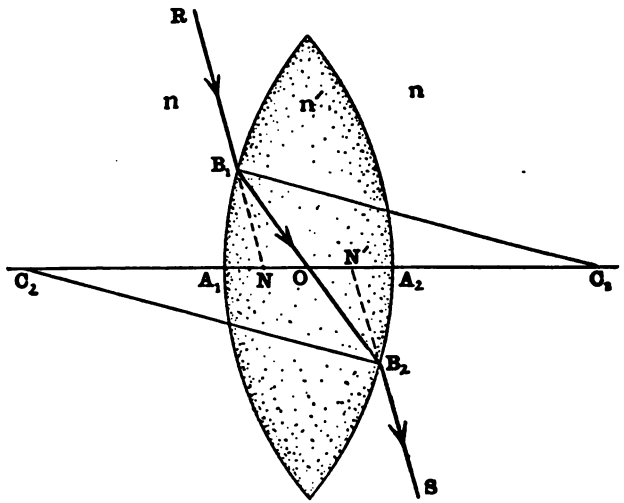


FIG. 116.—Optical center of lens.

geometrical form of the lens. In particular, the position of this point will not depend on the direction of the incident ray, as will be shown by the following investigation. From the similar triangles OC_1B_1 and OC_2B_2, we derive the proportion:

$$OC_1 : OC_2 = B_1C_1 : B_2C_2 = A_1C_1 : A_2C_2.$$

Accordingly, we may write:

$$\frac{OA_1 + A_1C_1}{OA_2 + A_2C_2} = \frac{A_1C_1}{A_2C_2},$$

and, consequently:

$$\frac{A_1O}{A_2O} = \frac{r_1}{r_2}.$$

Now $A_2O = A_2A_1 + A_1O = A_1O - d$; so that we obtain finally:

$$A_1O = \frac{r_1}{r_1 - r_2} d.$$

The function on the right-hand side of this equation depends only on the form of the lens, so that the position of the

point O with respect to the vertex of the first face of the lens may be found immediately as soon as we know the magnitudes denoted by r_1, r_2 and d.

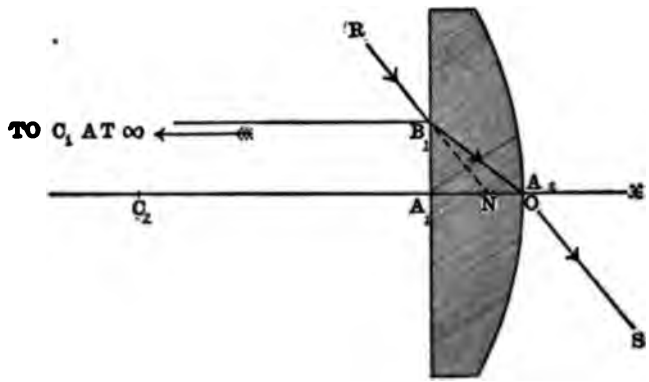


FIG. 117.—Optical center of lens with one plane face is at the vertex of curved face.

If the lens is double convex or double concave, the optical center O will lie inside the lens between the vertices A_1 and

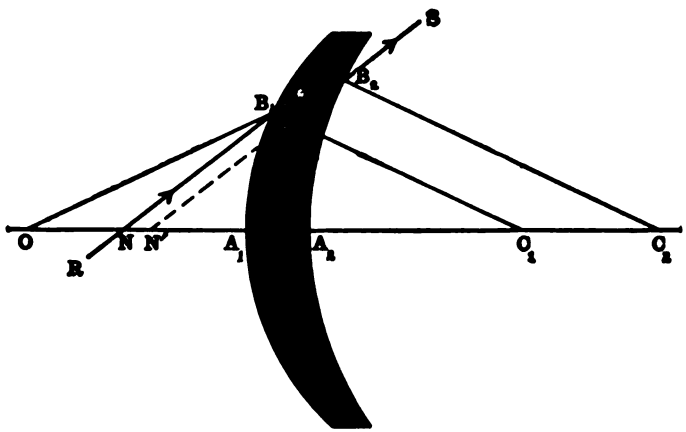


FIG. 118.—Optical center of meniscus lies outside lens.

A_2; if one face of the lens is plane (Fig. 117), the optical center will coincide with the vertex of the curved face; and,

finally, if the lens is a meniscus (Fig. 118), the optical center will lie outside the lens entirely.

In general, the positions of the points designated in the diagrams by the letters N, N′ will vary for different ray-paths B_1 B_2 within the lens; but if the rays are paraxial, the positions of N, N′ are fixed. In fact, if the ray RB_1 B_2 S is a paraxial ray, the points N, N′ are the so-called nodal points of the lens (see § 119).

89. The Abscissa-Formula of a Thin Lens, referred to the axial point of the lens as origin.—Ordinarily, the axial thickness of a lens is much smaller than either of the radii of curvature, so that in many lens-problems this dimension is negligible in comparison with the other linear dimensions that are involved. Moreover, the lens-formulæ are greatly simplified by ignoring the thickness of the lens. However, in using these formulæ one must be duly cautious about taking too literally results that are strictly applicable only to an *infinitely thin lens*, whose vertices are regarded as coincident, that is, $A_1 A_2 = d = 0$. The approximate formulæ that are obtained for lenses of zero-thickness are often of very great practical utility, especially in the preliminary design of an optical instrument composed, it may be, of several lenses whose thicknesses are by no means negligible.

The optical center O of an infinitely thin lens coincides with the two vertices A_1, A_2, and hereafter these three coincident points in which the axis meets an infinitely thin lens will be designated by the simple letter A. An infinitely thin lens is represented in a diagram by the segment of a straight line which is bisected at right angles by the axis of the lens; the actual form of the lens being indicated by assigning the positions of the centers C_1, C_2 of the two faces. In order to tell at a glance the character of a lens, the form of it at the edges may be indicated, as shown in Fig. 119. Fig. 119,*a* is a conventional representation of an infinitely thin convex lens, and Fig. 119, *b* is a similar diagram for an infinitely thin concave lens.

Let us assume that the lens is surrounded by the same medium on both sides; and let n denote the index of refrac-

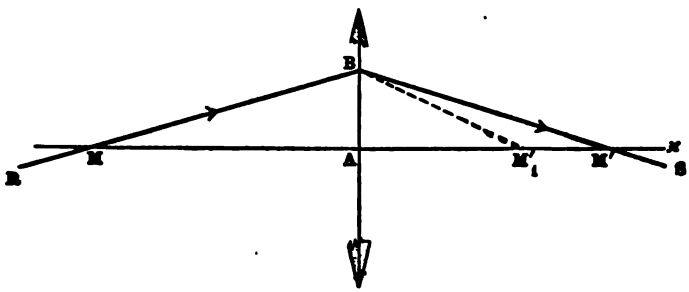


Fig. 119, *a*.—Infinitely thin convex lens; M, M′ conjugate points on axis.

tion of this medium, while n' denotes the index of refraction of the lens-substance itself.

The broken line RBS (Fig. 119) represents the path of a paraxial ray which enters and leaves the infinitely thin

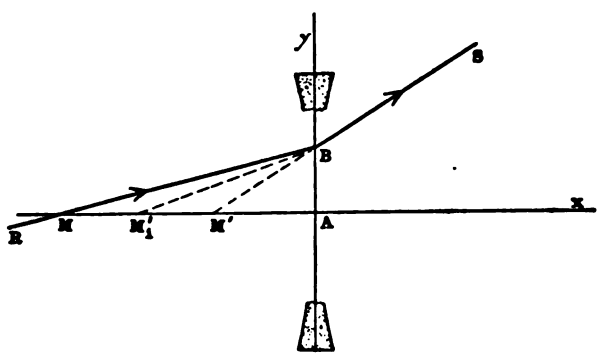


Fig. 119, *b*.—Infinitely thin concave lens; M, M′ conjugate points on axis.

lens at the point marked B. The points where the ray crosses the axis before and after passing through the lens will be designated by M, M′, respectively. The straight

line BM_1' which intersects the axis at the point marked M_1' shows the path the ray takes after being refracted at the first face of the lens. Obviously, the points M, M_1' are a pair of conjugate axial points with respect to the first surface of the lens, and, similarly, the points M_1', M′ are a pair of conjugate axial points with respect to the second surface of the lens, and, therefore, M, M′ are a pair of conjugate axial points with respect to the lens as a whole, so that M′ will be the image in the lens of an axial object-point M. The abscissæ of these points with respect to the axial point A as origin will be denoted by u, u'; thus, $u=AM$, $u'=AM'$. Also, put $u_1'=AM_1'$. The radii of curvature of the two faces are $r_1=AC_1$, $r_2=AC_2$.

Accordingly, in order to obtain the formulæ connecting u and u', we have merely to apply the fundamental equation (§ 78) for the refraction of paraxial rays at a spherical surface to each face of the lens in succession, bearing in mind that the first refraction is from medium n to medium n', while the second refraction is from medium n' to medium n. Thus, we obtain:

$$\frac{n'}{u_1'}-\frac{n}{u}=\frac{n'-n}{r_1}, \qquad \frac{n}{u'}-\frac{n'}{u_1'}=-\frac{n'-n}{r_2}.$$

Eliminating u_1' by adding these equations, and dividing through by n, we derive the *abscissa-formula for the refraction of paraxial rays through an infinitely thin lens*, in the following form:

$$\frac{1}{u'}-\frac{1}{u}=\frac{n'-n}{n}\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

The expression on the right-hand side of this equation, involving only the lens-constants r_1, r_2 and n'/n, has for a given lens a perfectly definite value, which may be computed once for all. And so if we put

$$\frac{1}{f}=\frac{n'-n}{n}\left(\frac{1}{r_1}-\frac{1}{r_2}\right),$$

where the magnitude denoted by f is a constant of the lens

(which we shall afterwards see is the *focal length* of the lens), the formula above may be written:

$$\frac{1}{u'}-\frac{1}{u}=\frac{1}{f};$$

which is the form of the lens-formula that is perhaps most common. For a given value of u we find $u'=f.u/(f+u)$.

Incidentally, it may be observed that the equation above is symmetrical with respect to u and $-u'$; that is, the equation will remain unaltered if $-u$ is written in place of u' and $-u'$ in place of u. Accordingly, if the positions of a pair of conjugate points on the axis are designated by M, M'

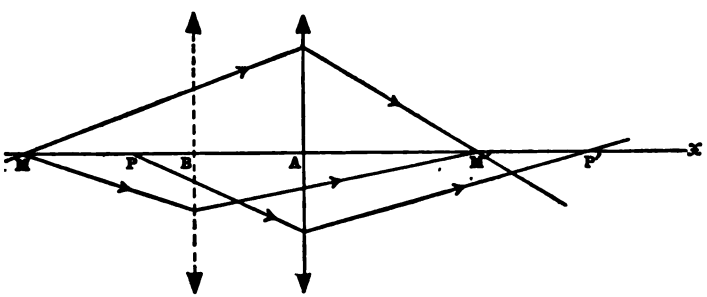


FIG. 120.—Infinitely thin lens: AP = M'A = BM, AP' = MA.

(Fig. 120), the pair of axial points designated by P, P' will likewise be conjugate, provided AP = M'A and AP' = MA; so that the thin lens at A bisects the two segments PM' and P'M. Another and more striking way of exhibiting this characteristic property of an infinitely thin lens consists in saying, that if M' is the image of an axial object-point at M, and if then the lens is shifted from its first position at A to a point B such that MB = AM', the object-point M will again be imaged at M'.

90. The Focal Points of an Infinitely Thin Lens.—If the object-point M is at the infinitely distant point on the axis of the lens, its image will be formed at a point F' whose position on the axis may be found by putting $u=\infty$, $u'=AF'$

in the formula $1/u' - 1/u = 1/f$; thus, we find $AF' = f$. Similarly, the object-point F conjugate to the infinitely distant point of the axis is found by substituting in the same equation the pair of values $u = AF$, $u' = \infty$, whence we obtain $AF = -f$. These points F, F′ are *the primary and secondary focal points*, respectively, and, accordingly, it is evident that *the focal points of an infinitely thin lens are equidistant from the lens and on opposite sides of it.*

The character of the imagery in the case of an infinitely thin lens is completely determined as soon as we know the positions of the two focal points F, F′; and since the point A where the axis meets the lens lies midway between F and F′, it is obvious that the natural division of lenses is into two classes depending on the order in which the three points above mentioned are ranged along the axis.

(1) *If the primary focal point is in front of the lens* (Fig. 121, *a*), that is, if the order of the points named in the sequence in which they are reached by light traversing the axis of the lens is F, A, F′, then incident rays parallel to

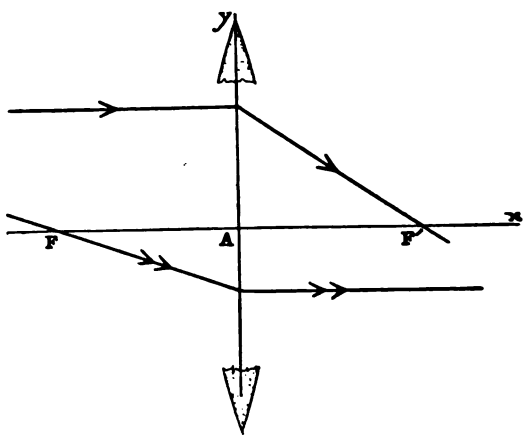


FIG. 121, *a*.—Focal points (F, F′),of infinitely thin lens (FA = AF′ = *f*). In a positive (or convex or convergent) lens the first focal point (F) lies on same side of lens as incident light (real focus).

the axis will be converged to a real focus at F′ on the other side of the lens, and the lens is a *convergent lens* (§ 87). It is also called a *positive lens*, because the lens-constant (or *primary focal length*) $f = FA = AF'$ is measured along the axis in the positive sense. If it is assumed that $n' > n$ (as, for example, in the case of a glass lens in air), the sign of this constant f, according to the formula above which defines $1/f$, will be the same as that of the term $(1/r_1 - 1/r_2)$, which is the algebraic expression of the difference of curvatures (§ 99) between the two faces of the lens. If the lens is double convex, plano-convex or a crescent-shaped meniscus—that is, in all forms of lenses that are thicker in the middle than out towards the edges—the difference of curvatures $(1/r_1 - 1/r_2)$ will be found to be positive. And hence, as already stated (§ 87), thin lenses of this description are convergent if $n' > n$.

(2) *If the secondary focal point is in front of the lens* (Fig. 121, *b*), that is, if the points F′, A, F are ranged along the

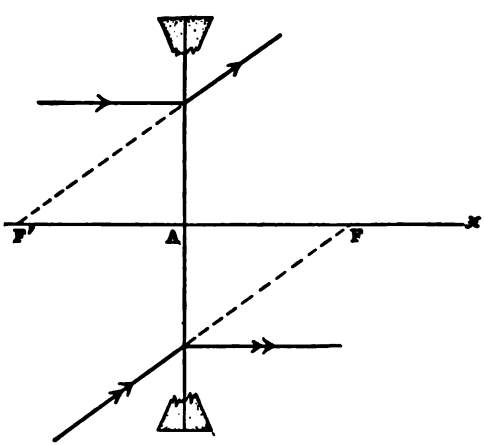


FIG. 121, *b*.—Focal points (F,F′) of infinitely thin lens ($FA = AF' = f$). In a negative (or concave or divergent) lens the first focal point (F) lies on the other side of the lens from the incident light (virtual focus).

axis in the order named, incident parallel rays will be made to diverge from a virtual focus at F′, and in this case the lens is said to be a *divergent* or *negative lens*, since now the lens-constant $f=FA=AF'$ is measured along the axis in the negative sense. For lenses which are thinner in the middle than at the edges, that is, for double concave, plano-concave and canoe-shaped meniscus lenses the difference of curvatures $(1/r_1-1/r_2)$ will be found to be negative; and hence for such lenses the constant f will be negative if $n'>n$.

A case of rather more theoretical than practical interest is afforded by an *infinitely thin concentric lens* (§ 87) for which $r_2=r_1$, and which is therefore uniformly thick in a direction parallel to the axis, so that according to the above classification it should be neither convergent nor divergent. In fact, the value of the lens-constant f for this lens is infinity, and hence $u'=u$, so that object-point M and image-point M′ are coincident always. A bundle of parallel rays traversing an infinitely thin concentric lens will emerge from the lens just as though the lens had not been interposed in the path of the rays.

91. Construction of the Point M′ Conjugate to the Axial Point M with respect to an Infinitely Thin Lens.—The planes which are perpendicular to the axis of the lens at the focal points F, F′ are called the *primary* and *secondary focal planes*, respectively.

The point M′ conjugate to a point M on the axis of an infinitely thin lens surrounded by the same medium on both sides may be constructed as follows:

Through the given point M (Fig. 122, *a* and *b*) draw a straight line MB meeting the lens at B, and through the axial point (A) of the lens draw a straight line AI′ parallel to MB and meeting the secondary focal plane in the point I′; then the point where the straight line BI′, produced if necessary, crosses the axis will be the required point M′ conjugate to M.

The point M′ may also be constructed in another way,

as follows: Let J designate the point where the straight line MB crosses the primary focal plane, and through B draw a straight line parallel to the straight line JA, which

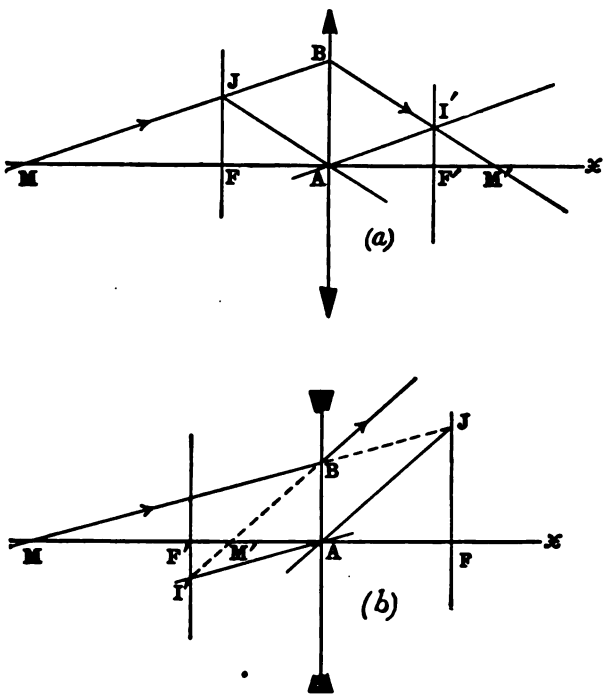


FIG. 122, *a* and *b*.—Infinitely thin lens: Construction of point M′ conjugate to axial object-point M. (*a*) Convex, (*b*) Concave lens.

will intersect the axis of the lens in the required point M′. Fig. 122, *a* shows the construction in the case of a convex lens and Fig. 122, *b* shows it for a concave lens.

The proof is obvious. From the two pairs of similar triangles MAB, AF′I′ and MM′B, AM′I′, we obtain the proportions:

$$\frac{MA}{AF'}=\frac{MB}{AI'}=\frac{MM'}{AM'};$$

and if we introduce the symbols $u = AM$, $u' = AM'$, $f = AF'$, we get:

$$\frac{-u}{f} = \frac{u'-u}{u'};$$

which is the same as the abscissa-relation found in § 89.

92. Extra-Axial Conjugate Points Q, Q'; Conjugate Planes.—Since the axial point A of an infinitely thin lens is also the optical center of the lens (§ 89), a straight line drawn through A will represent the path of a ray both before and after passing through the lens at this point. If the axis of the lens is rotated in a meridian plane through

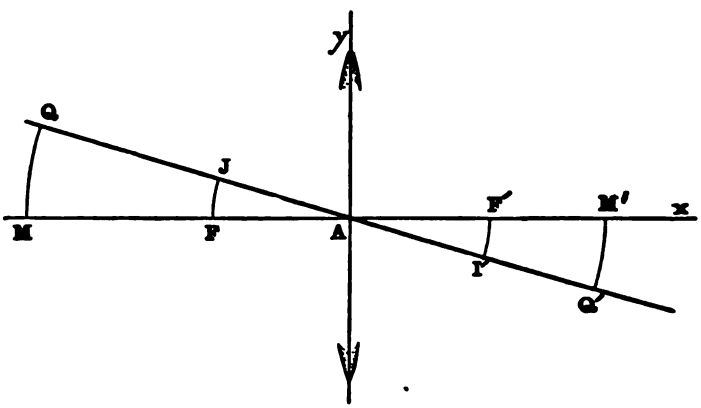


Fig. 123.—Infinitely thin lens: Image-point Q' conjugate to extra-axial object-point Q.

a very small angle FAJ (Fig. 123) around the point A as vertex, the focal points F, F' will describe the small arcs FJ, F'I' and the straight line JI' will represent the path of a paraxial ray traversing the lens at A. The points Q, Q' at the ends of the arcs MQ, M'Q' traced out in this angular movement of the axis by a pair of conjugate axial points M, M' will evidently occupy the same relation to each other on the straight line JI' as M, M' have to each other

on the straight line FF′, and therefore Q, Q′ are a pair of *extra-axial conjugate points.*

Accordingly, if the points of an object lie in the vicinity of the axis on an element of a spherical surface described

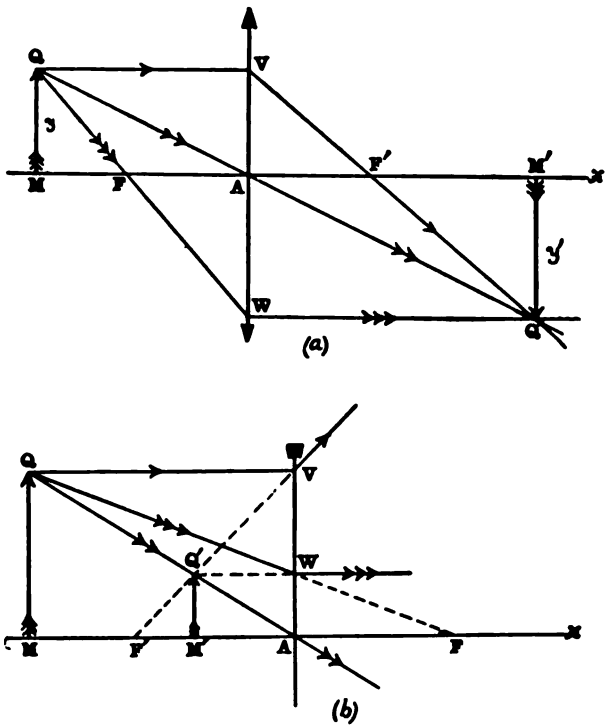


FIG. 124, *a* and *b*.—Infinitely thin lens: Lateral magnification and construction of image M′Q′ conjugate to short object-line MQ perpendicular to axis. (*a*) Convex, (*b*) Concave lens.

around the vertex A of the infinitely thin lens as center, the corresponding points of the image will be assembled on a concentric spherical surface; and since, within the region of paraxial rays, these spherical elements may be regarded as plane, it follows that a small plane object at

right angles to the axis will be reproduced by a similar plane image also at right angles to the axis.

Conjugate planes are pairs of parallel planes perpendicular to the axis of the lens; and any straight line drawn through the center of an infinitely thin lens will pierce a pair of conjugate planes in a pair of conjugate points.

In particular, the planes conjugate to the focal planes are the infinitely distant planes of the image-space and object-space, according as the infinitely distant plane is regarded as belonging to one or the other of these regions.

The construction of the point Q′ conjugate to an extra-axial object-point Q (Fig. 124, *a* and *b*), with respect to an infinitely thin lens, is made by a method precisely similar to that employed in the corresponding problem in the cases both of a spherical mirror (§ 71) and of a spherical refracting surface (§ 81); the only difference in this case being that the center of the lens takes the place of the center of the spherical surface and that the focal points of the lens are at equal distances on opposite sides of it.

93. Lateral Magnification in case of Infinitely Thin Lens.—The lateral magnification in the case of an infinitely thin lens, defined, as in §§ 72 and 82, as the ratio of the height of the image ($y' = M'Q'$) to the height of the object ($y = MQ$), may be obtained from the diagram (Fig. 124) and is evidently given by the following formula:

$$m = \frac{y'}{y} = \frac{u'}{u};$$

so that *the linear dimensions of object and image are in the same ratio as their distances from the thin lens.* Moreover, it appears that *the image is erect or inverted according as object and image lie on the same side or on opposite sides of the lens.*

Another expression for the lateral magnification may be derived by considering the two pairs of similar right

triangles FMQ, FAW and F′M′Q′, F′AV, from which we obtain the proportions:

$$\frac{AW}{MQ}=\frac{FA}{FM},\qquad \frac{M'Q'}{AV}=\frac{F'M'}{F'A};$$

and since

$$AW=M'Q'=y',\qquad AV=MQ=y,\qquad FA=AF'=f,$$

we find:

$$y=\frac{y'}{y}=\frac{f}{x}=-\frac{x'}{f},$$

where $x=FM$, $x'=F'M'$ denote the abscissæ of M, M′ with respect to the focal points F, F′, respectively, as origins. Accordingly, *the lateral magnification varies inversely as the distance of the object from the primary focal plane, and directly as the distance of the image from the secondary focal plane.*

94. Character of the Imagery in a Thin Lens.—The Newtonian form of the abscissa-relation (*cf.* § 85) for an infinitely thin lens surrounded by air is:

$$x.x'=-f^2,$$

which shows that object and image lie on opposite sides of the focal planes; so that if M is a point on the axis to the right of the primary focal point F, the conjugate point M′ will be found on the axis at the left of the secondary focal point F′, and *vice versa.*

The character of the imagery produced by the refraction of paraxial rays through an infinitely thin lens is exhibited in the diagrams Fig. 125, *a* and *b*. The numerals 1, 2, 3, etc., mark the successive positions of an object-point which is supposed to traverse a straight line parallel to the axis (so-called "object-ray") from an infinite distance in front of the lens to an infinite distance on the other side of it. Until it reaches the lens at the point marked V the object is real, thereafter it is virtual. The corresponding numerals with primes, viz., 1′, 2′, 3′, etc., ranged along the straight line VF′ (called the "image-ray") mark the successive positions of the image-point, which, starting, from the

secondary focal point F′, moves along this line always in the same direction out to infinity and back again to its starting point. The straight lines 11′, 22′, 33′, etc., con-

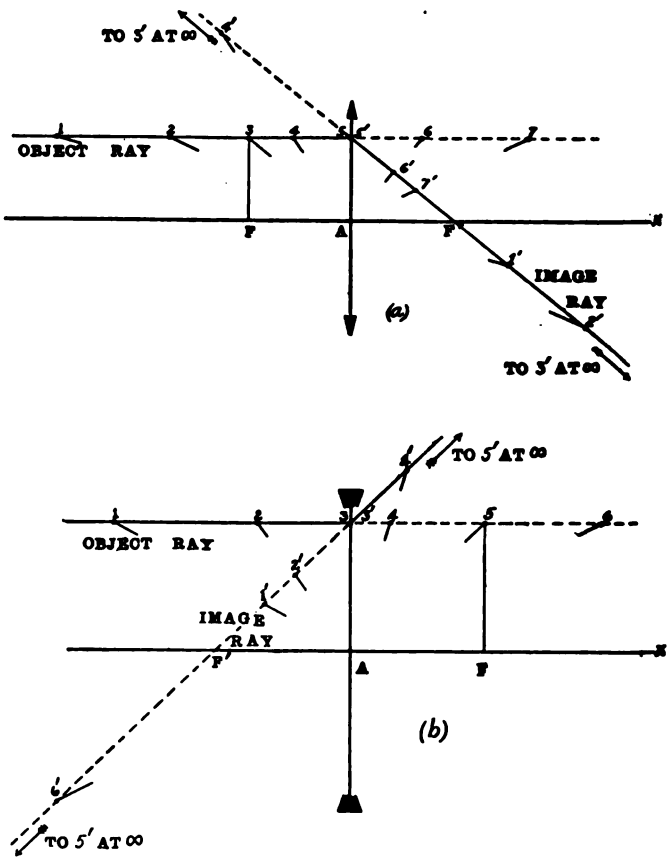


FIG. 125, *a* and *b*.—Character of imagery in infinitely thin lens. (*a*) Convex, (*b*) Concave lens.

necting corresponding positions of object-point and image-point form a pencil of rays all passing through the optical center A of the lens, which is the center of perspective of

object-space and image-space. At the point V object and image coincide with each other in the lens itself, and here object and image are congruent. The so-called *principal planes* (§ 119) *of an infinitely thin lens coincide with each other in the plane perpendicular to the axis of the lens at its optical center* A. The fact that object-point and image-point coincide with each other at V is expressed geometrically by saying *the y-axis is the base of a range of self conjugate points.*

In a convex lens (Fig. 125, *a*) the image of a real object is seen to be real and inverted as long as the object lies in front of the lens beyond the primary focal plane; whereas the image is virtual and erect if the object is placed between the primary focal plane and the lens. The image of a virtual object in a convex lens is formed between the lens and the secondary focal plane and is real and erect.

In a concave lens (Fig. 125, *b*) the image of a real object lies between the lens and the secondary focal plane, and it is virtual and erect. If the object is virtual, its image in a concave lens will be real and erect if the object lies between the lens and the primary focal plane, but it will be virtual and inverted if the object lies beyond the primary focal plane.

If $z = MM'$ denotes the distance between a pair of conjugate axial points M, M', then $u' = u + z$, where $u = AM$, $u' = AM'$. Substituting this value of u in the formula $1/u' - 1/u = 1/f$, we obtain a quadratic in u, which implies, therefore, that for a given value of the interval z between object and image, there are always *two* positions of the object-point M with respect to the lens (§ 89). But under some circumstances the assigned value of the interval z may be such that the roots of the quadratic prove to be imaginary, and then it will be quite impossible with the given lens to produce an image at the given distance z from the object. For example, if the object lies in front of a *convex lens* ($f > 0$) at a distance greater than the focal length,

then $u<0$ and $z>0$. Put $a=\mathrm{MA}=-u$, so that the magnitudes denoted by f, z and a are all positive. Eliminating u' from the abscissa-formula, we obtain a quadratic in a whose roots are given by the following expression:

$$a=\frac{z\pm\sqrt{z(z-4f)}}{2};$$

which will be imaginary if $(z-4f)<0$. Hence, *the distance* (z) *between a real object and its real image in a convex lens cannot be less than four times the focal length* f.

95. The Focal Lengths f, f′ of an Infinitely Thin Lens.—The focal lengths of a thin lens are defined exactly in the same way as the focal lengths of a spherical refracting sur-

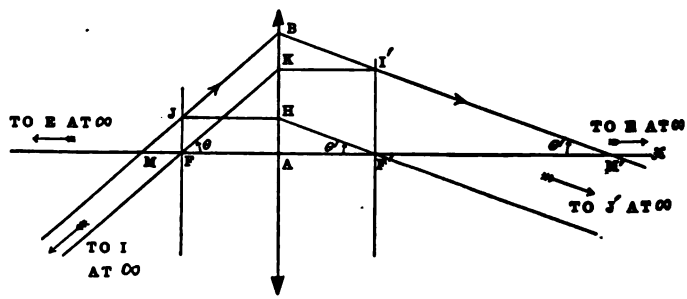


FIG. 126.—Focal planes and focal lengths of infinitely thin lens $(f=\mathrm{FA}=-f'=\mathrm{AF}')$.

face (§ 83). Thus, *the primary focal length of a lens is the ratio of the height of the image, in the secondary focal plane, to the apparent size of the infinitely distant object.* In Fig. 126 F′I′ is the height of the image of the infinitely distant object EI which is seen under the angle $\theta=\angle\mathrm{EFI}=\angle\mathrm{AFK}$, and the primary focal length is, therefore, $\mathrm{F'I'}/\tan\theta=\mathrm{AK}/\tan\theta=\mathrm{FA}=f$; and hence, as already observed, the primary focal length is identical with the lens-constant denoted by f, which, as we have seen (§ 90), is the abscissa of the axial point A of a thin lens with respect to its primary focal point F. Similarly, *the secondary focal length* (f') *is*

the ratio of the height of an object in the primary focal plane of the lens to the apparent size of its infinitely distant image. For example, in the diagram the image of the object FJ lying in the primary focal plane is E'J', which lying in the infinitely distant plane of the image-space, subtends the angle $\theta' = \angle EF'J' = \angle AF'H$; and hence $f' = FJ/\tan\theta' = AH/\tan\theta' = F'A$; so that the secondary focal length may also be defined as the abscissa of the axial point A of an infinitely thin lens with respect to the secondary focal point F'. And since $F'A = -AF' = -FA$, evidently:

$$f' = -f.$$

Accordingly, *the focal lengths* (f, f') *of a lens surrounded by the same medium on both sides are equal in magnitude and opposite in sign.*

If the lens is reversed by turning it through 180° about an axis perpendicular to the axis of the lens, that is, if the light is made to traverse the lens in a sense exactly opposite to that which it had at first, the focal lengths f, f' will not be altered. This is evident from the fact that the expression for the focal length f, viz.,

$$f = -f' = \frac{n.r_1.r_2}{(n'-n)\ (r_2-r_1)},$$

remains the same when $-r_1$, $-r_2$ are substituted in place of r_1, r_2, respectively. Thus, the character of the lens (§ 90) and its action are not changed by presenting the opposite face to the incident rays.

The focal length of an infinitely thin symmetric lens (§ 87), for which $r_1 = -r_2 = r$ (say) is $f = \frac{n.r}{2(n'-n)}$; and if $n = 1$, $n' = 1.5$, we find $f = r$. Accordingly, *the focal length of an infinitely thin symmetric glass lens surrounded by air* $(n = 1,\ n' = 1.5)$ *is equal to the radius of the first face.* Spectacle glasses were at first symmetric lenses, and in the old inch system of designation a No. 10 spectacle glass, for example, was a lens whose radius of curvature on each surface was 10 inches and whose focal length was 10 inches.

If one face of the lens is plane, for example, if $r_1=\infty$, $r_2=r$, we find $f=-\frac{n.r}{n'-n}$; or if $r_1=r$, $r_2=\infty$, then $f=\frac{n.r}{n'-n}$, where in each case r denotes the radius of the curved surface. Comparing this with the value of f obtained in the preceding case, we see that *if one of the faces of a symmetric lens be ground off plane, the focal length of the lens will thereby be doubled.*

96. Central Collineation of Object-Space and Image-Space.—Comparing the methods and results of this chapter with those obtained in the preceding chapter, the serious student cannot have failed to remark a striking parallelism that exists between the imagery by paraxial rays in a spherical refracting surface and the imagery under the same conditions in an infinitely thin lens. In some instances the formulæ are actually identical, and a closer examination will show that this similarity extends even to comparative details. For, example, the focal points lie on opposite sides of a lens just as they were found to do in the case of a spherical refracting surface, and the resemblance goes still farther. For in a spherical refracting surface the connection between the focal lengths (f, f') and the indices of refraction (n, n') is expressed by the formula $n'.f+n.f'=0$ (§ 79); and if in this formula we put $n'=n$, we obtain the relation $f+f'=0$, which is the algebraic statement of the fact that the focal lengths of a lens surrounded by the same medium on both sides are equal and opposite (§ 95).

It has already been pointed out that the imagery in a spherical mirror may be regarded as a special case of refraction at a spherical surface (§§ 75, 77 and 78); and now it is proposed to advance a step farther in this generalization process and to show that all these types of imagery which have been investigated separately and independently are in reality embraced in a concept of geometry known as *collinear correspondence* between one space and another (called in the theory of optical imagery "object-space"

and "image-space"). Moreover, these types of imagery belong to a particularly simple kind of collinear correspondence to which the name *central collineation* has been given.

A lens or an optical instrument is said to divide the surrounding space into two parts, viz., the object-space and the image-space; but these are not to be thought of as separate and distinct regions but as interpenetrating and including each other; so that a point or ray may be regarded at one time as belonging to the object-space and at another time as belonging to the image-space, depending merely on the point of view. Thus, for example, the infinitely distant plane of space may be viewed as the image of the primary focal plane of a lens, and then it is a part of the image-space; but if the secondary focal plane is regarded as the image of the infinitely distant plane, the latter is a part of the object-space.

Now the distinguishing characteristics of the optical imagery which is produced by the refraction of paraxial rays at a single spherical surface or through an infinitely thin lens may be summarized in the two following statements:

(*a*) *All straight lines joining pairs of conjugate points intersect in one point,* viz., the center (C) of the spherical refracting surface or the optical center (A) of the thin lens. This point which is the center of perspective of object-space and image-space is called the *center of collineation,* and will be referred to here as the point C.

(*b*) *Any pair of corresponding incident and refracted rays lying in a meridian plane meet in a straight line* A*y called the axis of collineation* (or the y-axis) *which is perpendicular at* A *to the optical axis* (or the x-axis).

Any straight line going through the center of collineation is called a *central ray. Every central ray is a self-corresponding ray;* that is, image-ray and corresponding object-ray lie along one and the same straight line. Moreover, any point lying on the axis of collineation is a *self-conjugate*

point; that is, *along this line object-point and image-point are coincident with each other.* The center of collineation is also a self-conjugate point, and hence, *in general, there are two self-conjugate points on a central ray,* viz., the center of collineation and the point where the ray meets the axis of collineation. Only in case the center of collineation lies on the axis of collineation will there be only one self-conjugate or so-called double point on a central ray.

97. Central Collineation (cont'd). Geometrical Constructions.—Starting from these simple propositions, we can easily develop a complete theory of optical imagery for the simple cases mentioned above. Thus, for example,

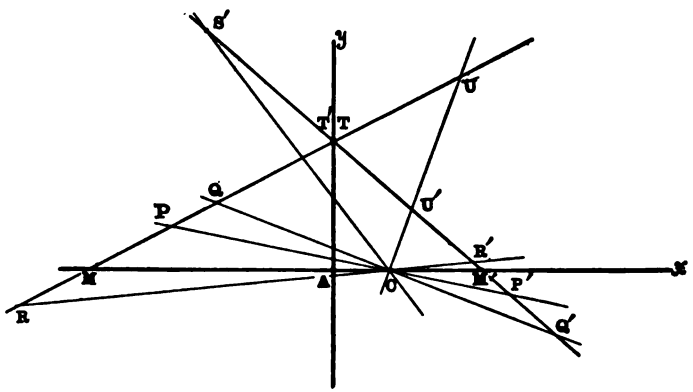


FIG. 127.—Central collineation: Construction of pairs of conjugate points M, M'; P, P'; Q, Q'; R, R'; S, S'; T, T'; and U, U'. Axis of collineation Ay; center of collineation C.

being given the axis of collineation (Ay) *and the center of collineation* (C), *together with the positions of a pair of conjugate points* P, P', *we can construct the position of a point* Q' *conjugate to a given point* Q, as follows:

(*a*) In general, the straight line PQ (Fig. 127) will not pass through the center of collineation. Let the self-conjugate point in which the straight line PQ meets the axis of collineation be designated by T; the image-ray cor-

responding to the object-ray PT will lie along the straight line TP′, and since this ray must pass likewise through the point Q′ conjugate to Q, the required point will be at the intersection of the straight lines TP′, QC.

(*b*) But in the special case when the straight line PQ is a central ray (Fig. 128) the construction which has just been given fails, and we must resort to a different procedure,

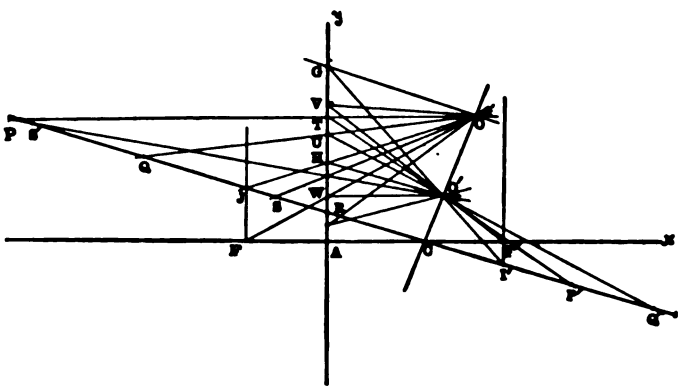


FIG. 128.—Central collineation: Straight line PQ passes through center of collineation (C). Diagram shows case when C does not lie on axis of collineation Ay; as in spherical refracting surface ($c > 1$).

as follows: Through the points P and C draw a pair of straight lines PO, CO meeting in a point O, and let the point where the straight line PO meets the axis of collineation be designated by T. Also, let O′ designate the point of intersection of the straight lines TP′ and CO. Then if the point where the straight line QO meets the axis of collineation is designated by U, the required point Q′ will be the point of intersection of the straight lines UO′ and QC.

The image-point I′ conjugate to the infinitely distant object-point I of the pencil of parallel rays whose central ray is PP′ may be constructed exactly as described above in (*b*), provided we have the same data. The straight line

OG is drawn parallel to PP′ meeting the axis of collineation in G, and the required point I′ is the point of intersection of the straight lines GO′ and PP′.

Similarly, the position of the object-point J conjugate to the infinitely distant image-point J′ of the central ray PP′ is found by drawing the straight line O′H parallel to PP′ meeting the axis of collineation in H; then the point of intersection of the straight lines OH, PP′ will be the required point J.

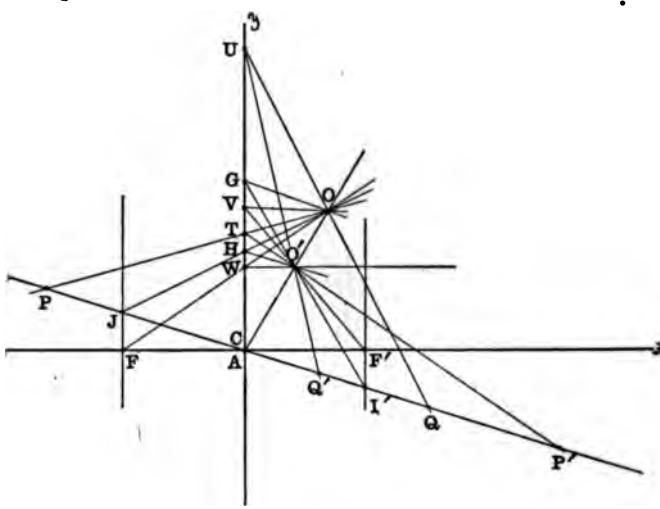


FIG. 129.—Central collineation: Straight line PQ passes through center of collineation (C). Diagram shows case when C lies on axis of collineation Ay, as in infinitely thin lens ($c=1$).

The focal points F, F′ on the optical axis are constructed in precisely the same way as the two points J, I′ on the central ray PP′.

The special case when *the center of collineation* (C) *lies on the axis of collineation*, that is, when the two points A and C are coincident, is shown in Fig. 129, which evidently corresponds to the case of an infinitely thin lens surrounded by the same medium on both sides.

It would be easy to show by the methods of projective geometry that the straight lines FJ, F′I′ are parallel to the axis of collineation and that we have the following relations between the points J, I′ and the two self-conjugate points B, C on the central ray JI′:

$$JB = CI', \quad I'B = CJ, \quad \frac{I'B}{BJ} = c,$$

where c denotes a constant called *the invariant of central collineation*, which has the value $n' : n$ for a spherical refracting surface and the value $+1$ for a thin lens surrounded by the same medium on both sides. For a spherical mirror, $c = -1$. For the axial ray the above relations may be written:

$$FA = CF', \quad F'A = CF, \quad \frac{F'A}{AF} = c.$$

The reader who wishes to pursue this subject will find a complete discussion at the end of Chapter V of the author's *Principles and Methods of Geometrical Optics* published by The Macmillan Company of New York.

98. Field of View of an Infinitely Thin Lens.—If it is assumed that there are no artificial stops present except in the plane of the lens, and that the imagery is produced by means of paraxial rays only, the field of view in the case of an observer looking through the lens along its axis is easily determined by drawing the straight lines O′G, O′H (Fig. 130, *a* and *b*) in a meridian plane of the lens from the center O′ of the eye-pupil to the ends G, H of the diameter of the lens-opening. For the lens-opening acts here just like a round window or port-hole in an opaque wall to limit the field of view in the image-space of the lens. If O designates the position of the axial object-point which is reproduced by the image-point O′, then the straight lines OG, OH determine the limits in the meridian plane of the diagram of the field of view of the object-space. Let the straight line B′C′ bisected at right angles at O′ by the axis of the lens represent the diameter of the pupil in the meridian plane of the lens; and construct the line BOC whose image in the lens

is the diameter B′O′C′ of the pupil of the eye. Then the image S′ of the luminous point S lying within the object-side field of view may be constructed by drawing through S the straight lines SB, SC to meet the lens in two points which

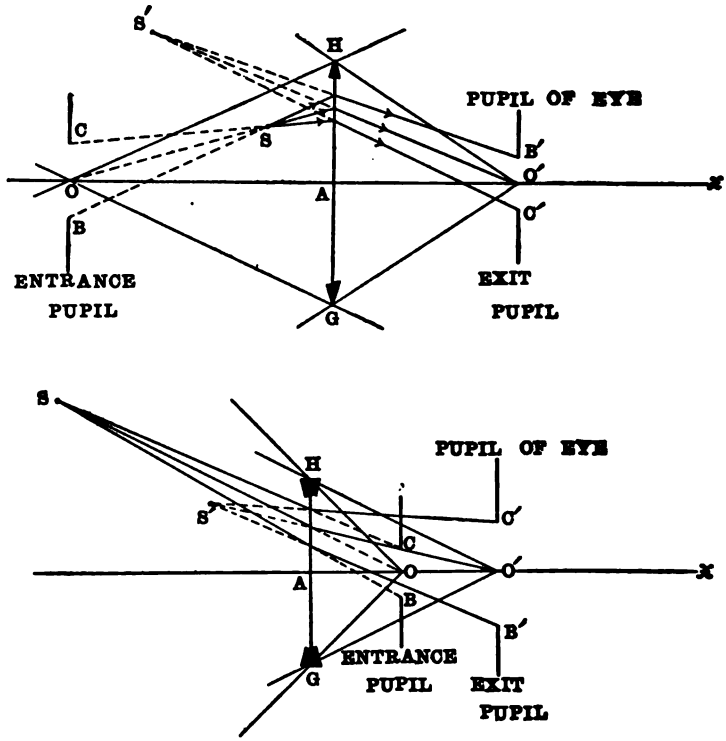


Fig. 130, *a* and *b*.—Field of view of infinitely thin lens for given position of eye on axis of lens. (*a*) Convex, (*b*) Concave lens.

must be joined with B′, C′, respectively; and the point of intersection of these latter lines will be the required point S′ conjugate to S. In brief, the circular opening whose diameter is BC is the common base of all the cones of effective rays in the object-space of the lens, just as the pupil of the eye itself is the common base of the cones of effective rays

in the image-space. Assuming that the lens-opening is large enough to permit the entire pupil of the eye to be filled with rays emanating from an axial object-point, the lens-opening GH acts as field-stop and the pupil of the eye as aperture-stop (Chapter XII).

PROBLEMS

1. Show how to construct the optical center of a lens. Draw diagrams for the various forms of convex and concave lenses; and prove that the distance of the optical center from the vertex of the first face is equal to $r_1 d/(r_1 - r_2)$, where r_1, r_2 denote the radii of the two surfaces and d denotes the axial thickness of the lens.

2. In each of the following lenses the axial thickness is 2 cm. Find the position of the optical center, and draw a diagram for each lens showing the position of this point.

(*a*) Double convex lens of radii 10 and 16 cm.; (*b*) Double concave lens of radii 10 and 16 cm.; (*c*) Plano-convex lens; (*d*) Positive meniscus of radii 10 and 16 cm.; (*e*) Negative meniscus of radii 20 and 16 cm.; (*f*) Lens of zero curvature.

3. Rays of light diverging from a point one foot in front of a thin lens are brought to a focus 4 inches beyond it. Find the focal length. Ans. $f = +3$ inches.

4. An object is placed one foot in front of a thin convex lens of focal length 9 inches. Where is the image formed? Ans. 3 feet from the lens on the other side.

5. Rays coming from a point 6 inches in front of a thin lens are converged to a point 18 inches on the other side of the lens. Find the focal length. Ans. $f = +4.5$ inches.

6. An object is placed in front of a thin lens at a distance of 30 cm. from it. The image is virtual and 10 cm. from the lens. Find the focal length. Ans. $f = -15$ cm.

7. The radius of the first face of a thin double convex lens made of glass of index 1.5 is 20 cm. If the focal length of the lens is 30 cm., what must be the radius of the second face? Ans. 60 cm.

8. A thin convex lens made of glass of index 1.5 has a focal length of 12.5 cm. If the radius of the second face is +17.5 cm., what is the radius of the first face? And if the lens is concave, and the radius of the first face is +17.5 cm., what is the radius of the second face?

Ans. In both cases the radius is +4.6 cm.

9. The focal length of a double convex lens was found to be 30.6 cm., and its radii 30.4 and 34.5 cm. Find the index of refraction of the glass. Ans. 1.528.

10. The focal length of a glass lens in air is 5 inches. What will be the focal length of the lens in water, assuming that the indices of refraction of air, glass and water are 1, $\frac{3}{2}$ and $\frac{4}{3}$, respectively? Ans. 20 inches.

11. Show that any thin lens which is thicker in the middle than out towards the edges is convergent, provided the lens-medium is more highly refracting than the surrounding medium.

12. Show that the focal length of a thin plano-convex lens is twice that of a double convex lens, if the curvatures of the curved surfaces are all equal in magnitude.

13. Find the focal length of a thin double convex diamond lens, of index 2.4875, the radius of each surface being 4 cm. Ans. 13.4 mm.

14. The curved surface of a thin plano-convex lens of glass of index 1.5 has a radius of 12 inches. Find its focal length. What must be the radii of a symmetric double convex lens of same material which has same focal length?

Ans. $f=24$ inches; $r=24$ inches.

15. The radii of a thin double convex lens are 9 cm. and 12 cm. The lens is made of glass of index 1.5. If light diverges from a point 18 cm. in front of the lens, where will it be focused? Ans. Real image, 24 cm. from lens.

16. A thin lens is made of glass of index n. If the focal length of the lens in air is a, and if its focal length in a liquid is b, show that the index of refraction of the liquid is

$$\frac{bn}{b+a(n-1)}.$$

17. Draw figures, approximately to scale, showing the paths of the rays of light, and the positions of the images formed when a luminous object is placed at a distance of (*a*) 1 inch, (*b*) 6 inches from a convex lens of focal length 2 inches.

18. An object is placed 8 inches from a thin convex lens, and its image is formed 24 inches on the other side of the lens. If the object were moved nearer the lens until its distance was 4 inches, where would the image be?

Ans. Virtual image, 1 foot from lens.

19. A virtual image of an object 30 cm. from a thin lens is formed on the same side of the lens at a distance of 10 cm. from it. Find the focal length of the lens.

Ans. $f = -15$ cm.

20. Light converging towards a point M on the axis of a lens is intercepted and focused at a point M′ on the same side of the lens as M. The distances of M and M′ from the lens are 5 cm. and 10 cm., respectively. Find the focal length of the lens. Ans. $f = -10$ cm.

21. A far-sighted person can see distinctly only at a distance of 40 cm. or more. How much will his range of distinct vision be increased by using spectacles of focal length +32 cm.?

Ans. The spectacles will enable him to see distinctly objects as near to his eye as 17.78 cm., so that his range of distinct vision will be increased by 22.22 cm.

22. The projection lens of a lantern has a focal length of one foot. If the screen is 1024 feet away, how far back of the lens must the glass slide be placed? Ans. 1024/1023 ft.

23. An engraver uses a magnifying glass of focal length +4 inches, holding it close to the eye. At what distance must the lens be from the work so that the magnification may be fourfold? Ans. 3 inches.

24. Assuming that the optical system of the eye is equivalent to a thin convex lens of focal length 15 mm., what will

be the size of the retinal image of a child 1 meter high at a distance of 15 meters from the eye? Ans. 1 mm.

25. A millimeter scale is placed at a distance of 84 cm. in front of a convex lens, and it was found that 10 mm. of the scale corresponded to 29 mm. of its real inverted image. Find the focal length of the lens. Ans. $f=+62.5$ cm.

26. If X, X′ and Y, Y′ are two pairs of conjugate points on the axis of an infinitely thin lens, and if the lens is midway between X and Y′, show that it is also midway between X′ and Y.

27. M and M′ are a pair of conjugate axial points with respect to an infinitely thin lens whose optical center is at a point designated by A. Show that when the lens is shifted from A to a point B such that MB=AM′, the points M and M′ will be conjugate to each other with respect to the lens in this new position.

28. Given the positions of the focal points F, F′ of an infinitely thin lens, show how to construct the image-point M′ conjugate to an axial object-point M. Draw diagrams for convex and concave lenses.

29. At the optical center (A) of a thin lens erect a perpendicular to the axis of the lens, and take a point L on this perpendicular such that $AL=f$, where f denotes the primary focal length. Through A draw a line AP in such a direction that $\angle F'AP=45°$, where F′ designates the secondary focal point of the lens. Take a point M on the axis of the lens, and draw the straight line ML meeting the straight line AP in a point S. If M′ designates the foot of the perpendicular let fall from S to the axis of the lens, show that M, M′ are a pair of conjugate axial points. Draw two diagrams, one for a convex and the other for a concave lens.

30. Derive the image-equations in the case of an infinitely thin lens in the form: $1/u'=1/u+1/f$, $y'/y=u'/u$.

31. Show that the focal points of an infinitely thin lens are at equal distances on opposite sides of the lens.

32. A candle is placed at a distance of 2 meters from a wall, and when a lens is placed between the candle and the wall at a distance of 50 cm. from the candle, a distinct image of the latter is cast upon the wall. Find the focal length of the lens and the magnification of the image.

Ans. $f=37.5$ cm.; image is 3 times as large as object.

33. The distance between a real object and its real image in an infinitely thin lens is 32 inches. If the image is 3 times as large as the object, find the position and focal length of the lens.

Ans. The lens is a convex lens of focal length 6 inches placed between object and image at a distance of 8 inches from the object.

34. When an object is held at a distance of 6 cm. from one face of a thin lens, the image of the object formed by reflection in this face is found to lie in the same plane as the object. If the object is placed at a distance of 20 cm. from the lens, the image produced by the lens is inverted and of the same size as the object. The lens is made of glass of index 1.5. Find the radii of the two surfaces.

Ans. The lens is a convex meniscus of radii 6 and $\frac{30}{11}$ cm.

35. In a magic lantern the image of the slide is thrown upon a screen by means of a thin convex lens. Show that the adjustment for focusing is always possible provided that the distance from the slide to the screen is not less than 4 times the focal length of the lens, and provided that the lens can move in its tube to a distance from the slide equal to twice the focal length.

36. A person holds a lens in front of his eye and observes that by reflection at the nearer surface an object which is 6 feet from the lens appears upright and diminished to one-twentieth of its height. Looking through the lens at an object on the other side 6 feet from the lens, its image is inverted and diminished in height to one-tenth. The lens is a glass lens of index 1.5. Find the radii of its surfaces. Ans. A double convex lens of radii $\frac{12}{19}$ and $\frac{12}{35}$ ft.

37. How far from a lens must an object be placed so that its image will be erect and half as high as the object?

Ans. The object must be in the second focal plane of the lens. (Draw diagram showing construction of image for convex lens and also a diagram for concave lens.)

38. How far from a thin lens must an object be placed so that its image will be inverted and half as high as the object? Draw two diagrams, showing construction of image for convex lens and for concave lens.

Ans. If the optical center of the lens and the primary focal point are designated by A and F, respectively, and if the axial point of the object is designated by M, then $AM = 3AF$.

39. An object is to be placed in front of a convex lens of focal length 18 inches in such a position that its image is magnified 3 times. Find the two possible positions, and draw diagram for each position showing the construction of the image.

Ans. If image is inverted, object must be 2 ft. from lens; if it is erect, object must be 1 ft. from lens.

40. In the preceding example if the lens were concave, where would the object have to be?

Ans. The object would be virtual, at a distance of 1 ft. from the lens for an erect image, and at a distance of 2 ft. for an inverted image.

41. A person can see distinctly at a distance of 1 foot, and he finds that when he holds a certain lens close to his eye small objects are seen distinctly and magnified 6 times. Find the focal length of the lens. Ans. $f = +2.4$ inches.

42. Derive the Newtonian formula $x.x' = -f^2$ for a lens.

43. A convex lens is used to produce an image of a fixed object on a fixed screen. Show that, in general, there will be two possible positions of the lens, and prove that the height of the object is the geometrical mean between the heights of the two images.

44. A copper cent is 19 mm. in diameter and a silver

half dollar is 30.4 mm. in diameter. How far from a convex lens of focal length 10 cm. must the smaller coin be placed so that its image in the lens will be just the size of the larger one?

Ans. It must be placed in front of the lens at a distance of either 16.25 cm. or 3.75 cm.

45. What must be the radius of the curved surface of a thin plano-convex lens made of glass of index 1.5 which will give a real image of an object placed 2 cm. in front of the lens and magnified 3 times? Ans. 9 mm.

46. Find the magnification of a convex lens of focal length 0.2 inch for an eye whose distance of most distinct vision is 14 inches. Ans. 71 times.

47. An object is placed in front of a convex lens at a distance from it equal to 1.5 times the focal length. Find the linear magnification. If the object is removed to twice this distance, what will be the magnification? Ans. -2; $-\frac{1}{2}$.

48. An object 5 cm. high is placed 12 cm. in front of a thin lens of focal length 8 cm. Find the position, size and nature of the image (*a*) for a convex lens, and (*b*) for a concave lens; and draw accurate diagram for each case.

Ans. (*a*) Real, inverted image, 10 cm. high, 24 cm. from lens; (*b*) Virtual, erect image, 2 cm. high, 4.8 cm. from lens.

49. When an object is placed at a point R on the axis of a thin lens of focal length f, the image is erect, and when the object is moved to a point S the image is the same size as before but inverted; show that

$$\mathrm{SR} = \frac{2f}{m},$$

where m is a positive number denoting the value of the ratio of the size of the image to that of the object.

50. A screen, placed at right angles to the axis of a thin lens of focal length f, receives the image of a small object. If the image is 20 times as large as the object, show that the distance of the screen from the lens is equal to $21f$.

51. Given a convex lens, a concave lens, a concave mirror and a convex mirror, each of focal length 20 cm. An object is placed in front of each in turn at distances of 40, 20 and 10 cm. Draw diagrams showing the construction of the image for each lens and each mirror and for each of the three given positions of the object; and find the position and character of the image in each case.

52. A plane mirror is placed anywhere behind a convex lens with its plane at right angles to the axis of the lens. A needle is set up perpendicular to the axis in the primary focal plane of the lens. Show that the image of the needle produced by rays that have passed twice through the lens will lie also in the primary focal plane and will be of the same size as the object but inverted.

53. An object is placed in front of a thin convex lens at a distance a from it not greater than twice its focal length f; and a plane mirror is adjusted in the secondary focal plane of the lens. Show that a real image formed by rays which have passed twice through the lens will be formed at a distance b in front of the lens; and that $f=(a+b)/2$. Show also that the image is of the same size as the object but inverted. Draw a diagram showing the construction of the image.

54. A convex lens of focal length 10 cm. is placed at a distance of 2 cm. in front of a plane mirror which is perpendicular to the axis of the lens. Where must an eye be placed in front of the lens so that it may see its own image by means of rays which, after having traversed the lens twice, return into the eye as bundles of parallel rays?

Ans. 3.75 cm. from the lens.

55. A thin convex lens of focal length 10 inches is placed in front of a concave mirror of focal length 5 inches. The distance between the lens and the mirror is 10 inches. An object is placed in front of the lens at any distance from it. Show that its image formed by rays which have passed twice through the lens will lie at an equal distance from the

lens on the other side of it, and that it will be of the same size as the object but inverted.

56. A thin convex lens of focal length 12 inches is placed 12 inches in front of a concave mirror of focal length 8 inches. An object is placed 3 inches in front of the lens. Show that its image formed by rays which have passed twice through the lens is in the same plane as the object and of the same size, but inverted.

57. The focal length of a thin symmetric double concave lens made of glass of index 1.5 is five inches. A luminous point lies on the axis so far away that it may be considered as being at infinity. Prove that its image formed by rays which are reflected at the first surface is 2.5 inches in front of the lens; the image formed by rays which are refracted twice at the first surface and reflected once at the second surface is on the other side of the lens at a distance of 1.25 inches from it; and, finally, the image formed by rays which after being reflected twice at the second surface have emerged again into the surrounding air is 0.5 inch from the lens on the side away from the source.

58. A concave mirror, of radius r, has its center at the optical center of a thin lens, of focal length f, and the axes of lens and mirror are in the same straight line. Rays coming from an axial object point at a distance u from the lens traverse the lens and after being reflected at the mirror pass through the lens again and emerge from it as a bundle of rays parallel to the axis. Prove that

$$\frac{1}{u}+\frac{2}{r}+\frac{2}{f}=0.$$

CHAPTER VIII

CHANGE OF CURVATURE OF THE WAVE-FRONT IN REFLECTION AND REFRACTION. DIOPTRY SYSTEM

99. Concerning Curvature and its Measure.—Since the rays or lines of advance of the light-waves are always at right angles to the wave-surface (§ 7), one way of investigating the procedure of light is to study the form of the wave-surface; for, in general, the effect of reflection and refraction will be to produce an abrupt change of curvature of the wave-front. In this method attention is concentrated primarily on the wave-surface rather than on the rays themselves; but in reality the only difference between it and the ray-method consists in a new point of view, which may, however, be serviceable. Thus, when a plane wave is incident on a lens, the wave-front on emergence will no longer be plane but curved in such fashion that the light-waves either converge to or diverge from a point in the second focal plane of the lens. The effect of the lens or optical system is to imprint a new curvature on the wave-front, and if the change of curvature which is thus produced can be ascertained, the final form of the wave can be determined by mere algebraic addition of the initial and impressed curvatures. It will be necessary, however, to explain precisely what is meant by this term curvature and how it is measured.

In passing along an arc of a plane curve from a point A (Fig. 131) to a point B, the *total curvature* of the arc AB is the change of direction of the curve between A and B, which is evidently measured by the angle between the tangents to the curve at these two places. This angle is equal to the angle at O between the normals AO and BO which are

perpendicular to the tangents at A and B. The *mean curvature* between A and B is the change of this angle per unit length of the arc AB. If, therefore, the length of the arc AB is denoted by a and the magnitude of the angle BOA

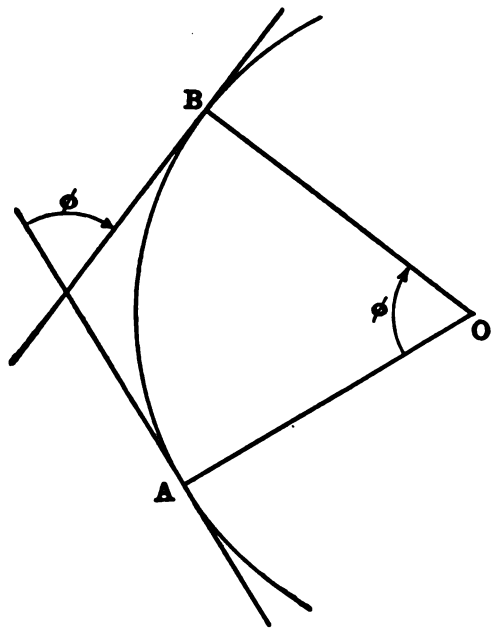


FIG. 131.—Mean curvature of arc AB measured by φ/a, where a denotes length of arc and ϕ denotes angle between the normals AO and BO.

by φ, the mean curvature between A and B is equal to φ/a. And the limiting value of this quotient when the point B is infinitely near to A is the measure of the actual curvature at the point A or, as we say, *the curvature at* A. If the curvature at A is denoted by the capital letter R, then R is equal to the limiting value of φ/a when the arc a is indefinitely small.

In Fig. 132 the point B is supposed to be infinitely near to A; and the point of intersection C of the normals drawn

to the two contiguous points A, B on the curve passi... through these two points is called the *center of curvatu...* of the curve at the point A; the circle described in th... plane of the curve around this point C as center with radi...

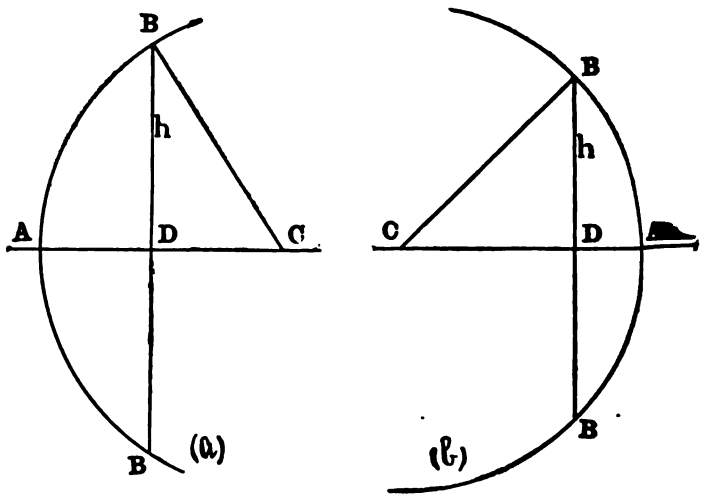


FIG. 132, *a* and *b*.—Curvature of arc BAB at point A midway betwe... B and B is measured by the sagitta AD. (*a*) Convex, (*b*) Concave arc...

$r = AC$, which will coincide with the given curve through...ut the infinitely small arc AB, is called the *circle of curvat...re* and its radius r is called the *radius of curvature* at the po...int A. Now since by definition the angle φ is equal to the arc BA divided by the radius r, that is, since $\varphi = a/r$, the curvature at A is equal to $1/r$; that is, *the curvature at any point on a curve is equal to the reciprocal of the radius of curvature at that point*, or

$$R = \frac{1}{r}.$$

The sign of the curvature is the same as that of the radius of curvature. Accordingly, *if the surface is convex with respect to the incident light, the curvature is to be counted as positive*, in accordance with our previous usage in this respect.

Thus, for example, when spherical waves spread out from a point-source, the wave-front at any instant is concave and its curvature is reckoned, therefore, as negative. If a convex lens is interposed at a distance from the point-source greater than its focal length, the light-waves will thereby be converged to a focus on the other side of the lens whence they will ultimately diverge again. While the wave-front is advancing from the lens to the focus, its curvature is positive; at the focus itself the wave-front collapses into a point, the curvature of the wave at this place being infinite; and beyond the focus the curvature becomes negative. As long as the wave does not undergo any reflection or refraction, its curvature varies continuously; whereas a sudden change of curvature is imprinted on the wave when there is a transition from one medium to another.

Another method of measuring the curvature of a small arc BB (Fig. 132) is in terms of its bulge AD, where the points designated by A and D are the middle points of the arc and its chord. If the points A and B are so close together that they may be regarded as lying on the circle of curvature corresponding to the point A, the ordinate $DB = h$ will be a mean proportional between the two segments into which the diameter of the circle is divided by the point D, so that we have the proportion:

$$AD : h = h : (2r - AD).$$

Since the segment AD is always very small in comparison with the diameter of the circle of curvature, only a vanishingly small error will be introduced by writing $2r$ in place of $(2r - AD)$ in the above proportion. Thus, we obtain:

$$AD = \frac{h^2}{2r},$$

or since $R = 1/r$,

$$AD = \frac{h^2}{2} R.$$

If the arc BB is not infinitely small, this equation contains a certain error which is more and more negligible in pro-

portion as the arc is taken smaller and smaller. For a small arc, therefore, we may say that the segment AD is proportional to the curvature (R) at the point A, and hence it may be said to measure the curvature at this place. This segment AD was called by KEPLER the *sagitta* of the arc BB because of its resemblance to an "arrow" on a bow.

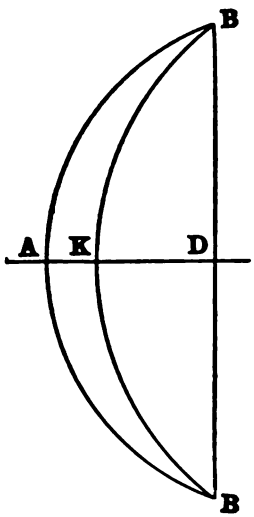


FIG. 133.—Curvatures of arcs BAB and BKB are in same ratio as their sagittæ AD and KD.

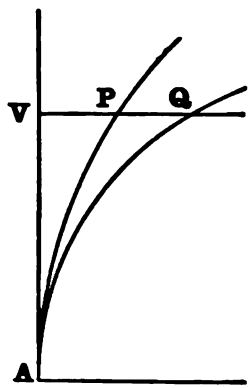


FIG. 134.—Curvatures of arcs AP and AQ in same ratio as their sagittæ VP and VQ.

Obviously, it does measure the bulge or "sag" of the curve at A. In Fig. 133, where the straight line BDB is the common chord of the small arcs BAB and BKB, the curvatures at A and K are evidently in the ratio of AD to KD. Or, again, consider Fig. 134, where the two arcs AP and AQ have a common tangent at A. If on this tangent a point V is taken very close to A, and if through V a straight line is drawn perpendicular to AV intersecting the two arcs in

the points designated by P and Q, the curvatures at A will be in the ratio of VP to VQ.

In many optical problems (as has been explained in the last two chapters) we are concerned only with a very small portion of the reflecting and refracting surface (case of paraxial rays), and under such circumstances it is especially convenient and simple to measure the curvatures of the wave-fronts before and after refraction or reflection and the curvatures of the mirrors or lenses by means of their sagittæ. In fact, the ordinary method of determining the curvature of an optical surface with an instrument called a *spherometer* (Fig.135) consists essentially in employing a micrometer screw to measure the *sagitta* of the arc whose chord is equal to the diameter of the circle circumscribed about the equilateral triangle formed by the conical points of the tripod which supports the instrument on the curved surface to be measured. The simple *lens-gauge* (Fig. 136) used by opticians to measure the power of a spectacle lens is based on the same principle. In size and external appearance it resembles a watch, except that on its lower side it has three metallic pins projecting from it in parallel lines which all lie in a plane parallel to the face of the gauge. The two outer pins are stationary and symmetrically placed so that when the instrument is held in a vertical plane with the pins pointing downwards, the straight line BB (Fig. 132) joining the conical points of the outer pins is horizontal; whereas

Fig. 135.—Spherometer.

the other pin which is midway between the two outer ones is capable of being pushed upwards by a slight pressure so that its tip A which left to itself falls a little below the straight line BB can be made to ascend a little above this line. The vertical displacement of the tip A of the middle pin above or below the level of the chord BB, which is equal to the *sagitta* of the arc BAB whose curvature is to be measured, is registered on the dial (see § 108) by the angular movement of a light hand or pointer with which the movable pin is connected. If the circle is drawn which passes through the end-points of the three pins B, A and B, the diameter drawn through A will bisect the chord BB at a point D; and since the products of the segments of two intersecting chords of a circle are equal, we obtain immediately:

FIG. 136.—Lens-gauge.

$$AD\,(2r - AD) = h^2,$$

where r denotes the radius of the circle and $2h$ = chord BB. Hence, exactly as above, we obtain here also:

$$AD = \frac{h^2}{2r}\text{, approximately;}$$

thus proving again that the *sagitta* AD is proportional to the curvature $1/r = R$. In using the lens-gauge care must be taken to see that the plane of the instrument is not tilted out of the vertical, and this is one reason why a spherometer is more accurate. On the other hand, the lens-gauge, besides being more handy and convenient, possesses a de-

cided advantage over a spherometer supported on a tripod by reason of the fact that it can be used to measure the curvatures in different meridians of a non-spherical surface of revolution, for example, the curvatures of the normal sections (§ 111) of a cylindrical or of a toric surface (§ 112). How the lens-gauge is graduated will be explained presently (§ 108).

100. Refraction of a Spherical Wave at a Plane Surface. —The whole duty of an optical system, therefore, whether

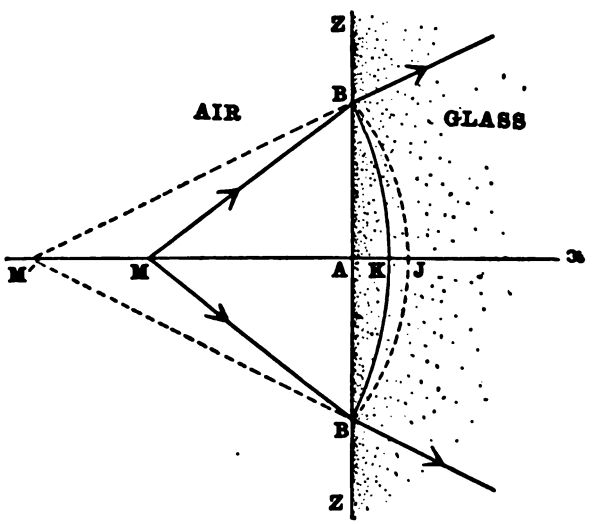


FIG. 137, *a*.—Divergent spherical waves refracted at plane surface from air to glass.

it be a single lens or mirror or a combination of such parts is to imprint a certain curvature on the surface of the incident wave; and if we consider only such portions of the wave-fronts as lie very close to the axis of symmetry of the instrument, it is evident that this method of investigating the change of curvature that is produced in the wave-front at the point where the axis meets it should lead to precisely the same results as have been found already

in the corresponding problems concerning the reflection and refraction of paraxial rays. In fact, according to this method, these results should be found to apply not merely to the case when the reflecting and refracting surfaces are plane or spherical, but equally also to the more general case when these surfaces have any form whatever, provided they are symmetrical around the optical axis.

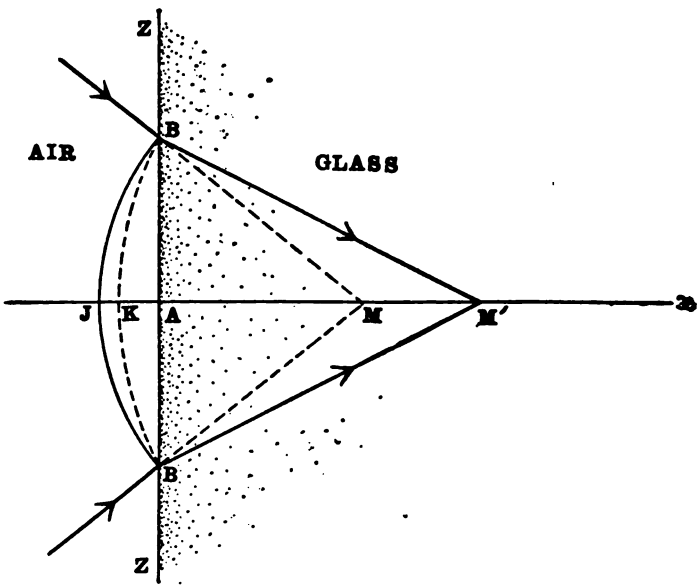


FIG. 137, *b*.—Convergent spherical waves refracted at plane surface from air to glass.

We shall begin by investigating the simple case of the refraction of a spherical wave at a plane surface.

In the diagrams (Fig. 137, *a*, *b*, *c*, and *d*) the straight line ZZ represents the trace in the plane of the paper of a plane refracting surface separating two media of indices n, n'. Around the point M as center spherical waves are supposed to be advancing in the first medium (n) towards the refracting surface, and at a certain instant when the disturbance

has begun to affect a point B on this surface the incident wave will be represented in the plane of the figure by the circular arc BJB described around M as center with radius equal to BM; the point designated by J lying on the arc midway between its two ends B, B, so that the straight line MJ is the perpendicular bisector at A of the chord BB. The two points M, J will be found to lie always on opposite

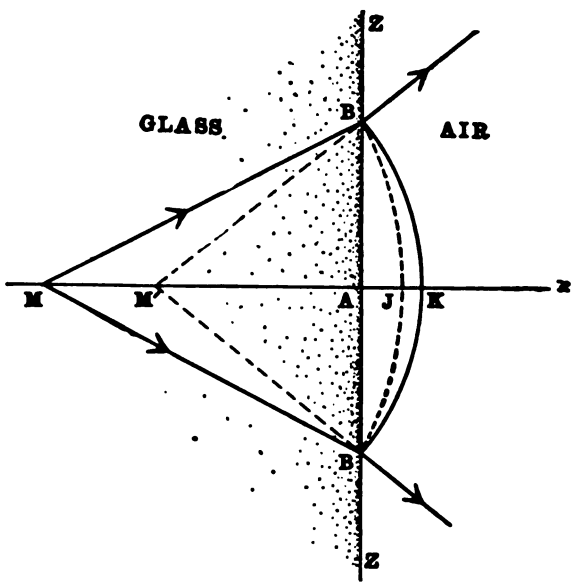


FIG. 137, c.—Divergent spherical waves refracted at plane surface from glass to air.

sides of the refracting plane. In Fig. 137, *a* and *c*, where the point M is shown as lying in front of the surface ZZ, the arc BJB is indicated by a dotted line, because it marks the position which the incident wave-front would have had if the refracting surface had not been interposed. But the waves travel faster in the rarer medium (air) than in the denser medium (glass); and, consequently, the vertex of the refracted wave-front instead of being at the point J

on the axis will be at a point K on this line, and therefore the position of the refracted wave-front at the moment when the disturbance arrives at B will be represented by the arc BKB of a circle whose center is at a point M' on the axis. If, for example, the waves are refracted from air to glass, that is, if $n'>n$, the velocity v in the first me-

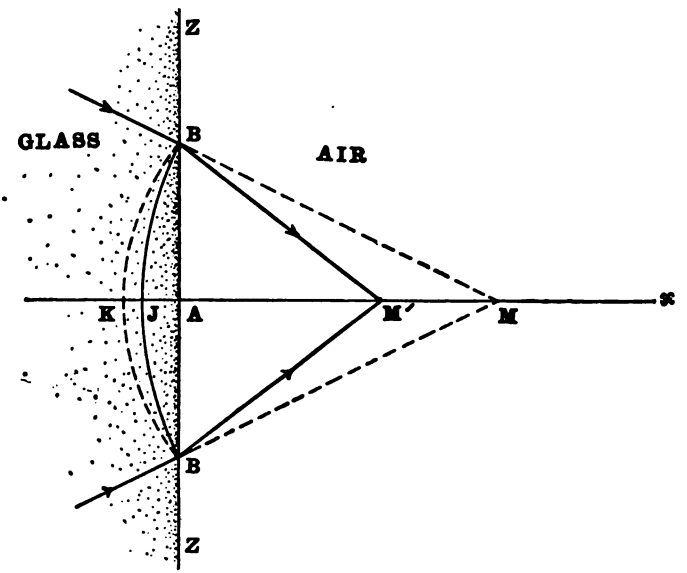


FIG. 137, *d*.—Convergent spherical waves refracted at plane surface from glass to air.

dium will be greater than the velocity v' in the second medium, so that for this case AK will be shorter than AJ, and the effect of the retardation will be to flatten the wave-front, as shown in Fig. 137, *a* and *b*. On the other hand, if $n'<n$, then $v'>v$, so that now AK will be longer than AJ, and the effect of the refraction will be to increase the curvature or bulge of the wave-front, as shown in Fig. 137, *c* and *d*. Since (see § 31)

$$AJ : AK = v : v' = n' : n,$$

it follows that

$$n'.KA = n.JA.$$

Now JA and KA are the *sagittæ* (§ 99) of the small arcs BJB and BKB, respectively, and hence they are proportional to the curvatures of these arcs, that is, to 1/JM and 1/KM'. If the point B is infinitely near to A, we may put $JM = AM = u$, $KM' = AM' = u'$; and thus we obtain:

$$\frac{n'}{u'} = \frac{n}{u};$$

which will be recognized as the relation which we found for the refraction of paraxial rays at a plane surface (§ 41).

101. Refraction of a Spherical Wave at a Spherical Surface.—Here the same method is employed as in the preceding section. In each of the diagrams (Fig. 138, *a*, *b*, *c*, *d*, *e*, *f*, *g*, and *h*) the circular arc ZZ represents the trace in the plane of the paper of a meridian section of the spherical refracting surface with its vertex at A and center at C. The surface is convex in Fig. 138, *a*, *b*, *c*, and *d* and concave in Fig. 138, *e*, *f*, *g*, and *h*. The point M on the axis is the center of a system of spherical waves which are advancing in the first medium, of index *n*, towards the refracting surface. In Fig. 138, *a*, *c*, *e*, and *g* the point M lies in front of the re-

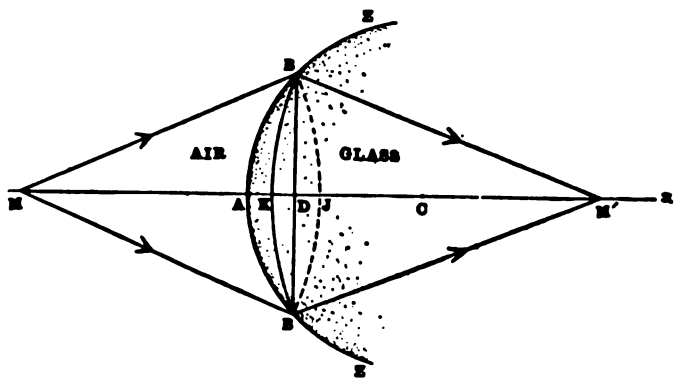


FIG. 138, *a*.—Divergent spherical waves refracted at convex surface from air to glass.

fracting surface, whereas in Fig. 138, *b*, *d*, *f*, and *h* this point is situated on the other side of the surface. The points

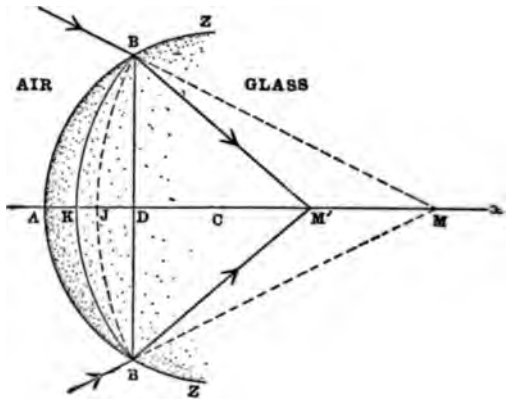


FIG. 138, *b*.—Convergent spherical waves refracted at convex surface from air to glass.

marked B, B are two points on the arc ZZ very close together but at equal distances on opposite sides of the op-

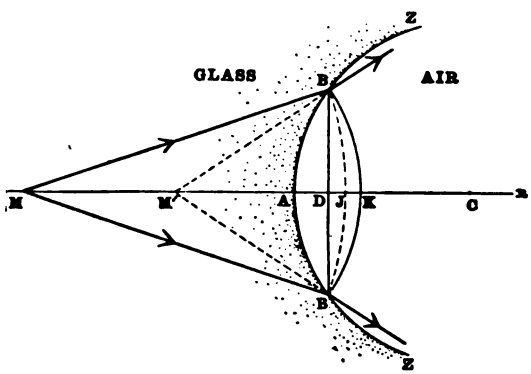


FIG. 138, *c*.—Divergent spherical waves refracted at convex surface from glass to air.

tical axis, so that the arc BJB described around M as center with radius equal to BM shows the position of the wave-front of the incident waves at the instant when the disturb-

ıce begins to affect the points B, B; the point where this
c crosses the optical axis being designated by J.

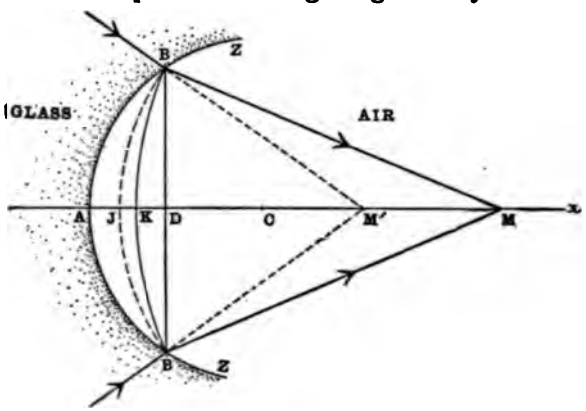


FIG. 138, *d*.—Convergent spherical waves refracted at convex surface from glass to air.

When the waves enter the second medium, of index n', ey will proceed with augmented or diminished speed ac-

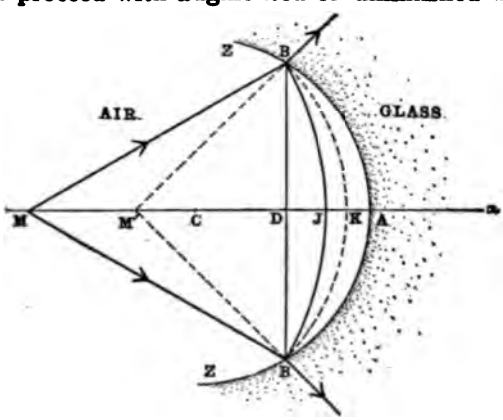


FIG. 138, *e*.—Divergent spherical waves refracted at concave surface from air to glass.

ırding as n is greater or less than n'. In the diagrams ig. 138, *a*, *b*, *e*, *f*, the case is represented where $n' > n$; and the diagrams Fig. 138, *c*, *d*, *g*, *h* the second medium is

supposed to be less highly refracting than the first ($n' <$ [illegible]). The center of curvature of the refracted waves will lie

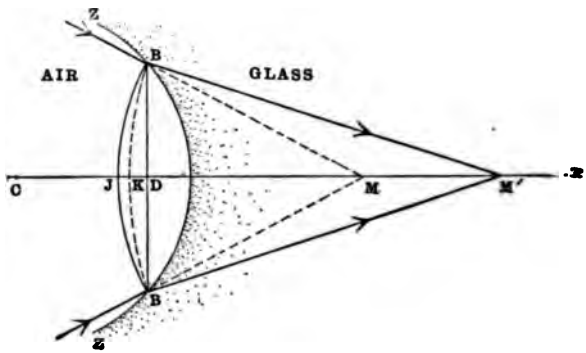


FIG. 138, *f*.—Convergent spherical waves refracted at concave surface from air to glass.

a point M′ on the axis, so that the wave-front in the second medium which passes through B, B will be represented by

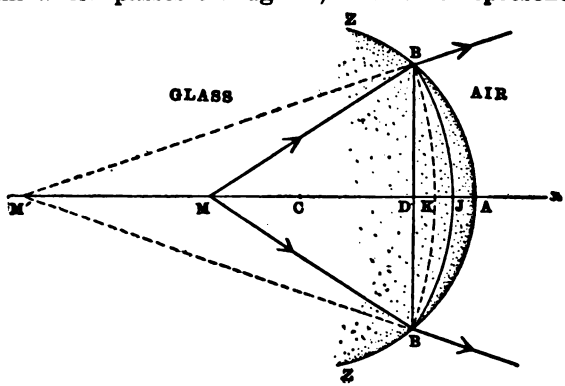


FIG. 138, *g*.—Divergent spherical waves refracted at concave surface from glass to air.

the arc BKB of a circle described around M′ as center with radius equal to BM′; the point where this arc crosses the axis being designated by K.

In each of the diagrams of Fig. 138 one of the two arcs

BJB and BKB is shown by a dotted line, because, on account of the interposition of the refracting surface ZZ, the part of one or the other of these wave-fronts which is comprised between B, B does not actually materialize; but this circumstance does not in the least affect the geometrical relations.

Thus, during the time the light takes to go in the first medium from J to A (or from A to J), it will travel in the

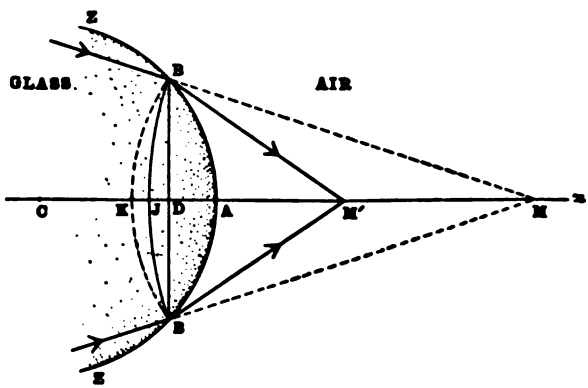


FIG. 138, A.—Convergent spherical waves refracted at concave surface from glass to air.

second medium from K to A (or from A to K). In other words, the optical lengths (§ 39) of the axial line-segments AJ and AK are equal, and therefore:

$$n.\mathrm{AJ} = n.\mathrm{AK}.$$

This shows how the position of the point M′ may be found, for we have only to lay off on the axis a piece

$$\mathrm{AK} = \frac{n}{n'}\mathrm{AJ},$$

and to locate the point M′ at the place where the perpendicular bisector of the chord BK intersects the optical axis.

Draw the chord BDB crossing the optical axis at right

angles at the point D; then, evidently, since

$$AJ = AD + DJ = AD - JD,\ AK = AD + DK = AD - KD,$$

we have: $$n(AD - JD) = n'(AD - KD).$$

Now recalling the fact that the points B, B were assumed to be very close to the vertex A of the spherical refracting surface, we remark that the arcs whose summits are at A, J and K are all very small; and hence the segments AD, JD and KD may be regarded as the *sagittæ* of these arcs and proportional to their curvatures (§ 99), viz., $1/r$, $1/u$ and $1/u'$, respectively, where $r = AC$, $u = AM = JM$, $u' = AM' = KM'$, approximately. Introducing these values in the equation above, we obtain the characteristic invariant relation for the case of the refraction of paraxial rays at a spherical surface, viz.,

$$n\left(\frac{1}{r} - \frac{1}{u}\right) = n'\left(\frac{1}{r} - \frac{1}{u'}\right),$$

in the same form as was found in § 78.

102. Reflection of a Spherical Wave at a Spherical Mirror.—The problem of reflection at a spherical mirror may

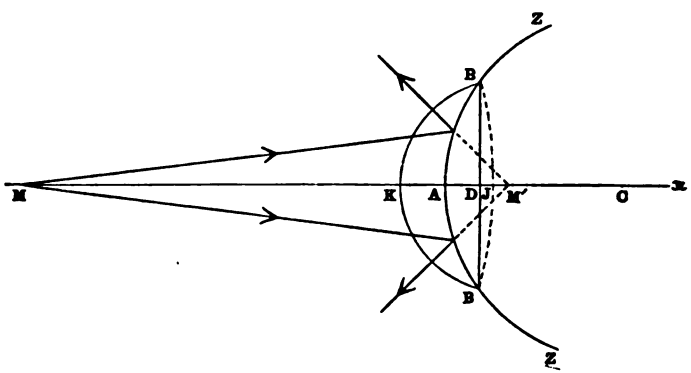


FIG. 139, *a*.—Divergent spherical waves reflected at convex mirror.

be investigated in the same way. In Fig. 139, *a* and *b*, the arcs BAB, BJB and BKB represent the traces of the mirror and of the wave-fronts of the incident and reflected

waves, respectively. In the case of reflection the condition evidently is:

$$KA = AJ,$$

because while the incident wave advances along the optical axis through the distance AJ or JA, the reflected wave will travel in the opposite direction through an equal distance

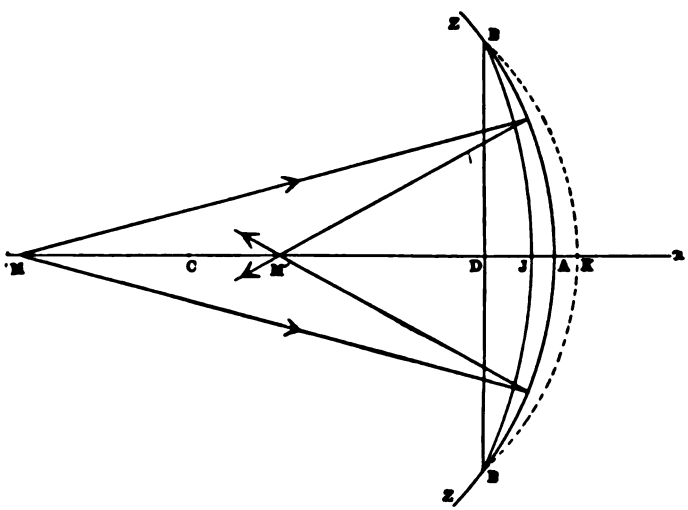


Fig. 139, *b*.—Divergent spherical waves reflected at concave mirror.

KA or AK. Therefore the center M′ of the reflected wave may be found by laying off AK = JA and locating the point where the perpendicular bisector of the chord KB intersects the axis.

Here also the segments AD, JD and KD are to be regarded as the measures of the curvatures of the small arcs BAB, BJB and BKB, respectively, and proportional, therefore, to the reciprocals of the radii of curvature, viz., $1/r$, $1/u$ and $1/u'$, where $r = AC$, $u = AM = JM$, $u' = AM' = KM'$ in the limit when the arcs are infinitely small. Now

$$KD = KA + AD = AJ + AD = AD + DJ + AD,$$

that is,

$$JD+KD=2AD;$$

hence, substituting the symbols u, u' and r, we derive the abscissa-formula for the reflection of paraxial rays at a spherical mirror (§ 64), viz.,

$$\frac{1}{u}+\frac{1}{u'}=\frac{2}{r};$$

which may be expressed in words by saying that *the curvature of the mirror is the arithmetical mean of the curvatures of the incident and reflected waves at the vertex of the mirror;* that is,

$$R=\frac{U+U'}{2},$$

where $U=1/u$, $U'=1/u'$ denote the curvatures of the incident and reflected waves, and $R=1/r$ denotes the curvature of the mirror. Thus, for example, if an incident plane wave ($U=0$) is advancing parallel to the axis of the mirror, the curvature of the reflected wave will be twice that of the mirror, and consequently, the center F of the reflected wave-front will lie midway between the vertex A and the center C of the mirror (§ 69).

Of course, the condition KA=AJ might have been derived at once from the condition $n.AJ=n'.AK$, which was found in § 101, by putting in this equation $n'=-n$, in accordance with the general rule given in § 75.

103. Refraction of a Spherical Wave through an Infinitely Thin Lens.—Since, as has been shown (§ 89), a homocentric bundle of incident paraxial rays with its vertex at a point M on the axis of a thin lens is transformed into a homocentric bundle of emergent rays with its vertex at the conjugate point M′, we know that if the waves are spherical before traversing the lens, they will issue from it as spherical waves, at least in the neighborhood of the axis.

Each of the diagrams (Fig. 140, *a* and *b*) represents a meridian section of the lens which is convex in one figure and concave in the other. As a matter of fact the lens is

assumed to be infinitely thin, and perhaps it is well to call particular attention to this fundamental consideration, because in the diagrams, in order to exhibit the relations by means of the *sagittæ*, the lens-thickness is shown very much exaggerated.

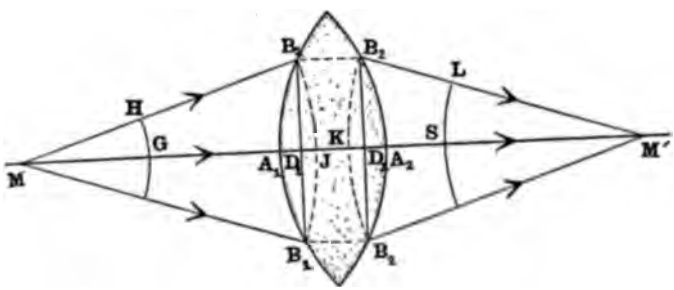


FIG. 140, *a*.—Divergent spherical waves refracted through thin convex lens.

Take a point B_1 on the first surface of the lens not very far from the vertex A_1 of this surface, and around the axial object-point M as center with radius equal to B_1M describe the circular arc B_1JB_1 which is bisected by the axis of the lens in the point designated by J; evidently, this arc will represent the trace in the plane of the diagram of the wave-front of the incident waves at the moment when the disturbance reaches B_1. Now the disturbance which is propagated onwards from B_1 will proceed across the lens to a point B_2 on the second face of the lens, and since the lens is supposed to be infinitely thin, the distances of B_1, B_2 from the axis are to be regarded as equal, that is, $D_1B_1 = D_2B_2$, where D_1, D_2, designate the feet of the perpendiculars let fall from B_1, B_2, respectively, to the axis of the lens. If, therefore, around the point M′ conjugate to M an arc B_2KB_2 is described with radius equal to B_2M', which is bisected by the axis at the point designated by K, this arc will represent the trace in the plane of the diagram of the wave-front of the emergent waves at the same instant

that the arc B_1JB_1 shows the wave-front of the incident waves.

With M, M′ as centers and with any convenient radii describe also the arcs GH, SL intersecting the axis of the

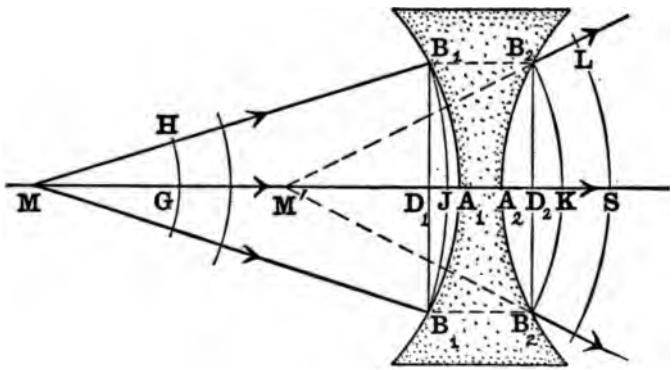


FIG. 140, *b*.—Divergent spherical waves refracted through thin concave lens.

lens at G, S and meeting the straight lines B_1M, B_2M', in H, L, respectively; so that these arcs represent, therefore, successive positions of the wave-front before and after transmission through the lens. Now the optical length of the light-path from H to L is equal to that along the axis of the lens from G to S (§ 39); and, hence, if n, n' denote the indices of refraction of the two media concerned, we may write:

$$n.HB_1+n'.B_1B_2+n.B_2L=n.GA_1+n'.A_1A_2+n.A_2S;$$

and since

$$n(MH+LM')=n(MG+SM'),$$

we obtain by addition of these two equations:

$$n(MB_1+B_2M')+n'.B_1B_2=n(MA_1+A_2M')+n'.A_1A_2.$$

Now $MB_1=MJ$, $B_2M'=KM'$, $B_1B_2=D_1D_2$;

and therefore:

$$n(MJ-MA_1+KM-A_2M')=n'(A_1A_2-D_1D_2).$$

Substituting in this equation the following expressions, viz.:

$$MJ-MA_1 = A_1M+MJ = A_1J = A_1D_1+D_1J = A_1D_1-JD_1,$$
$$KM'-A_2M' = KM'+M'A_2 = KA_2 = KD_2+D_2A_2$$
$$= KD_2-A_2D_2,$$
$$A_1A_2 = A_1D_1+D_1D_2+D_2A_2 = A_1D_1+D_1D_2-A_2D_2,$$

we obtain:

$$n(A_1D_1-JD_1+KD_2-A_2D_2) = n'(A_1D_1-A_2D_2);$$

which may be put finally in the following form:

$$n(KD_2-JD_1) = (n'-n)\ (A_1D_1-A_2D_2).$$

It has been assumed here that the lens is surrounded by the same medium (n) on both sides, but the same method would lead to a more general formula for which the initial and final media were different.

Evidently, since the points B_1, B_2 are very near the vertices A_1, A_2, the segments A_1D_1, JD_1, A_2D_2, KD_2 may be regarded as the *sagittæ* of the small arcs $B_1A_1B_1$, B_1JB_1, $B_2A_2B_2$, B_2KB_2, respectively; and since these arcs all have equal chords, the reciprocals of the radii of curvature may be substituted in the equation above in place of the *sagittæ*. Accordingly, if the radii of the lens-surfaces are denoted by r_1, r_2, and if we put $AM = JM = u$, $A_2M' = KM' = u'$, as is permissible in this case, we derive immediately the familiar lens-formula for the refraction of paraxial rays (§ 89), viz.:

$$n\left(\frac{1}{u'}-\frac{1}{u}\right) = (n'-n)\left(\frac{1}{r_1}-\frac{1}{r_2}\right) = \frac{n}{f},$$

where f denotes the primary focal length of the lens.

104. Reduced Distance.—If P, Q designate the positions of two points lying both in the same medium of refractive index n, a distinction has already been pointed out (see § 39) between the actual or absolute distance of these points from each other and the so-called "optical length" of the segment PQ of the straight line joining these points, which is obtained by multiplying the absolute length by the index of refraction of the medium, and which is equal therefore to $n.PQ$. A further distinction, due originally to Gauss,

is to be made now by employing the term *reduced distance* between P and Q to mean, not the product, but *the quotient of the distance* PQ *by the index of refraction of the medium in which the two points* P *and* Q *lie;* that is, the reduced distance from P to Q is equal to $\frac{PQ}{n}$. Thus, for example, if the medium is glass of index 1.5, and if the distance PQ= 12 inches, the optical distance or equivalent light-path in air will be 18 inches, whereas the reduced distance will be 8 inches. The reduced thickness of a lens is $c=\frac{d}{n}$, where $d=A_1A_2$ denotes the distance of the second vertex A_2 of the lens from the first vertex A_1 and n denotes the index of refraction of the lens-substance. The optical distance is never less, and the reduced distance is never greater, than the actual distance. If the medium is air ($n=1$), the optical distance and the reduced distance are both equal to the absolute distance. Apparently, the first use of the term "reduced distance" in this sense in English occurs in PENDLEBURY'S *Lenses and systems of lenses, treated after the manner of* GAUSS (Cambridge, 1884). A distinct advantage in the direction of simplification is usually gained in mathematical formulation by denoting a more or less complex function by a single symbol; and modern optical writers, notably GULLSTRAND and his disciples in Germany, have recognized the convenience of this idea of "reduced distance" and utilized it to express the relations between object and image in their simplest forms; as we shall show presently by several examples.

In this connection the attention of the student needs to be called to a point which has been alluded to before (see § 8), but which is not always clearly understood. Although two points P, Q may be situated physically in different media, they may be regarded as optically in the same medium. Thus, any point which is on the prolongation, in either direction, of the line-segment which represents the

actual path of a ray of light through a certain medium, *may*, and in fact generally *must*, be regarded as a point belonging to the medium in question, no matter what may be its actual physical environment. No better illustration of this notion can be given than is afforded by considering the focal points on the axis of a spherical refracting surface. The points F and F′ lie always on opposite sides of the vertex A, but no matter whether the first focal point F is on one side of A or on the other, it is to be considered always as a point in the first medium; and, similarly, the second focal point F′ is to be considered always as a point in the second medium, so that the reduced distance between F and F′ is $FA/n+AF'/n'$ both for a convergent and for a divergent system. The *reduced focal lengths* of a spherical refracting surface are $\frac{f}{n}$ and $\frac{f'}{n'}$; so that the reduced distance of F′ from F is equal to $\frac{f}{n}-\frac{f'}{n'}$.

The boundary between two optical media is a "twilight zone," so to speak, which cannot be said properly to belong to either medium; and hence linear magnitudes which refer specifically to the interface or surface of separation cannot be definitely assigned to one medium or the other. This applies, for example, to the radius of curvature of a mirror or of a refracting surface. Whether a surface which separates air from glass is convex or concave, we have no right to say that the radius of curvature lies in the air or in the glass; and thus we never speak of the "reduced radius" of a reflecting or refracting surface.

105. The Refracting Power.—In the u-form of the abscissa-equation which gives the relation between a pair of conjugate points on the axis, we are concerned not so much with the linear magnitudes themselves, that is, with the abscissæ, as with the reciprocals of these magnitudes, which, as we have seen, represent the curvatures of the surfaces of which these abscissæ are the radii. It is partly for this reason

that many teachers of geometrical optics regard the so-called "curvature method" of studying these problems as both more natural and more direct than the "ray method." There is certainly much to be said in its favor, but the truth is, both methods have their advantages, and neither is to be preferred to the other. The student who desires to have more than a mere elementary knowledge of optics will find it necessary to be acquainted with both points of view; and when he has attained this position, he will realize that the two methods are perfectly equivalent and that the distinction between them is more or less artificial.

But whether we have the so-called "curvature method" in mind or not, it will evidently be a step in the direction of simplifying the abscissæ-formula if we introduce symbols for the reciprocals of the abscissæ, and thereby get rid of the fractional forms. Thus, instead of employing the reduced focal length, it will be better to introduce a term for the reciprocal of this magnitude. Accordingly, the *refracting power* of an optical system is defined to be *the reciprocal of the reduced primary focal length.* These reciprocal magnitudes will be denoted by capital italic letters. For example, the refracting power of an optical system will be denoted by F; that is, according to the above definition:

$$F=\frac{n}{f}.$$

The refracting power of a *spherical refracting surface* (see § 79) is:

$$F=\frac{n}{f}=-\frac{n'}{f'}=(n'-n)R,$$

where $R=\frac{1}{r}$ denotes the curvature of the surface. If the first medium is air ($n=1$), then $F=\frac{1}{f}$. The refracting power of a spherical refracting surface is directly proportional to the curvature of the surface.

The *reflecting power* of a *spherical mirror* ($n' = -n$, $f' = f$) is defined in the same way, viz.,

$$F = \frac{n}{f} = -2n.R,$$

where n denotes the index of refraction of the medium in front of the mirror. Thus, although the position of the focal point (F) and the magnitude of the focal length (f) of a curved mirror will not be altered by changing the medium in front of the mirror, its reflecting power will be affected; and this will be the case whether the mirror is concave or convex. If the focal length of a mirror is 8, its reflecting power will be one-eighth when the mirror is in contact with air ($n = 1$), but it will be raised to one-sixth if the medium in front of the mirror is water ($n = \frac{4}{3}$).

The refracting power of a *lens* surrounded by the same medium (n) on both sides is

$$F = \frac{n}{f} = -\frac{n}{f'}.$$

If the curvatures of the two faces of an *infinitely thin lens* are denoted by R_1 and R_2, that is, if

$$R_1 = \frac{1}{r_1}, \quad R_2 = \frac{1}{r_2},$$

then

$$F = (n' - n)\ (R_1 - R_2),$$

where n' denotes the index of refraction of the lens-substance and n denotes the index of refraction of the surrounding medium. If either one of these media is changed, other things remaining the same, the refracting power of the lens will be altered.

If F_1, F_2 denote the refracting powers of the two surfaces of a lens, then

$$F_1 = (n' - n)R_1, \qquad F_2 = (n - n')R_2,$$

and in place of the preceding equation we may write:

$$F = F_1 + F_2;$$

and thus it appears that *the refracting power of an infinitely*

thin lens is equal to the algebraic sum of the refracting powers of the lens-surfaces.

The refracting power of a lens depends, therefore, on the curvatures of both faces, but evidently a lens of given material and of prescribed refracting power may have very different forms. One of the minor problems of optical construction is to *"bend" a lens*, as the technical phrase is, that is, being given the curvature of one face of the lens, to find the curvature of the other face so that the refracting power of the lens may have a given value. If, for example the magnitudes denoted by n, n', R_2 and F are assigned, the curvature of the first face must be:

$$R_1 = R_2 + \frac{F}{n'-n}.$$

If *the media are different on the two sides of the lens*, and if the indices of refraction of the three media in the order in which they are traversed by the light are denoted by n_1, n_2 and n_3, we find easily the following formula for the refracting power of an infinitely thin lens:

$$F = \frac{n_1}{f} = -\frac{n_3}{f'} = (n_2-n_1)R_1 + (n_3-n_2)R_2 = F_1 + F_2,$$

where the symbols have precisely the same meanings as before.

It will be seen from these examples that one effect of introducing the term refracting power is a simplification in consequence of the fact that the two magnitudes denoted by f and f' are now expressed in terms of a single magnitude F.

106. Reduced Abscissa and Reduced "Vergence".—The reduced abscissæ of a pair of conjugate axial points M, M′ are defined in exactly the same way as the reduced focal lengths. The point designated by M is to be regarded always as lying in the first medium of the system, and, similarly, the point designated by M′ is to be regarded as lying in the last medium, entirely irrespective of the question as

to whether either of these points is "real" or "virtual," as explained in § 104.

By way of illustration, suppose that the optical system consists of a single spherical refracting surface separating two media of indices n and n'. If the origin of abscissæ is taken at the vertex A, so that $u = \mathrm{AM}$, $u' = \mathrm{AM}'$, then the reduced abscissæ will be $\frac{u}{n}, \frac{u'}{n'}$. The reciprocals of these magnitudes, denoted by U, U' are called the *reduced "vergences,"* with respect to the point A; thus,

$$U = \frac{n}{u}, \qquad U' = \frac{n'}{u'}.$$

These functions U, U' are the measures of the convergence or divergence of the bundles of object-rays and image-rays; and in this illustration these magnitudes are evidently proportional to the curvatures of the incident and refracted wave-fronts at the instant when the disturbance arrives at the surface of separation of the two media.

Since (§ 79) the abscissa-formula for a *spherical refracting surface* may be written in the form:

$$\frac{n'}{u'} = \frac{n}{u} + \frac{n}{f},$$

this relation may now be expressed in the elegant and convenient form:

$$U' = U + F.$$

This same formula holds in the case of a *spherical mirror*, in which case $U' = -n/u'$, where n denotes the index of refraction of the medium in front of the mirror.

Moreover, the same formula $U' = U + F$ is found to be applicable to the case of an *infinitely thin lens*. If the lens is surrounded by the same medium (n) on both sides, then we must put $U = n/u$, $U' = n/u'$ and $F = n/f$, where n' denotes the index of refraction of the lens-substance. Or in case the last medium (n_2) is different from the first medium (n_1), then $U = n_1/u$, $U' = n_2/u'$, and $F = n_1/f$. In both cases the formula will be found to be identical in form with that

given above. In fact, as we shall see in Chapter X, the formula $U' = U + F$ is perfectly general and applicable to any optical instrument which is symmetrical about an axis. The advantage of a single formula which has such wide applicability is obvious. It is easy to remember that *the reduced vergence* (U') *on the image-side of the instrument is equal to the algebraic sum of the reduced vergence* (U) *on the object-side and the refracting power* (F).

If the abscissæ are measured from the focal points F, F', that is, if we put $x = \mathrm{FM}$, $x' = \mathrm{F'M'}$, the magnitudes

$$X = \frac{n}{x}, \quad X' = \frac{n'}{x'}$$

are called the *reduced focal point vergences;* and the relation between X, X' is expressed by the equation:

$$X \cdot X' = -F^2.$$

107. The Dioptry as Unit of Curvature.—Obviously, the magnitudes which have been denoted above by capital italic letters, since they are all equal or proportional to the reciprocals of certain linear magnitudes, are essentially measures of curvature, and hence they must be described or expressed in terms of some *unit of curvature*, which will itself be dependent on the unit of length. Opticians guided by purely practical considerations were the first to recognize the need of a suitable optical unit for this purpose. The unit of curvature which is now almost universally used in spectacle optics and which is coming to be employed more and more in all other branches of optics is the curvature of an arc whose radius of curvature is one meter. To this unit the name *dioptry** has been given. Originally, the

* The name "dioptrie" was first suggested by MONOYER of France in 1872 (see *Annales d'oculistique*, LXVIII, 111), being derived from the Greek τὰ διοπτρικά, whence came also the term "dioptrics" which was formerly much used by scientific writers as applying to the phenomena of refraction, especially through lenses. The word is generally written *dioptre* in French and *Dioptrie* in German. Etymologically, the correct English form would appear to be *dioptry*, and this spelling has been adopted by the American translators of both

dioptry was defined as the refracting power of a lens in air of focal length one meter. Consequently, a lens whose focal length was 50 cm. or half a meter would have a refracting power of 2 dptr., whereas another lens of focal length 2 meters would have a refracting power of $\frac{1}{2}$ dptr. In general, if the focal length of a lens surrounded by air is f centimeters, its refracting power will be $100/f$ dptr. But according to the definition which we have given, the dioptry is a unit not of refracting power only but of any similar magnitude of the nature of a curvature. Thus, for example, if the radius of a mirror or of a spherical refracting surface is half a meter, its curvature is 2 dptr. If the distances denoted by f, r, u, x, etc., are expressed in meters, the magnitudes denoted by the corresponding capital letters F, R, U, X, etc., will be in dioptries. Dr. Drysdale has suggested that we introduce also the convenient terms *millidioptry* (=0.001 dptr.), *Hectodioptry* (=100 dptr.) and *Kilodioptry* (=1000 dptr.) corresponding, respectively, to the Kilometer, centimeter and millimeter as units of length. Thus, the refracting power of a lens of focal length 10 cm. might be variously described as equal to 100 millidioptries, to 10 dioptries, to 0.1 Hectodioptry or to 0.01 Kilodioptry. But these terms have not come into general use.

If the focal length of a lens in water ($n=1.3$) is 13 cm., its refracting power will be the same as that of a lens in air ($n=1$) of focal length 10 cm., viz., 10 dptr. If the primary focal point of a spherical refracting surface is situated

Landolt's and Tscherning's books on physiological optics; notwithstanding the fact that the word is usually spelled and pronounced *dioptre* in England and *diopter* in America. Dr. Crew in his well known text-book of physics writes *dioptric*. The author has concluded that on the whole it is best to adopt the spelling used in the text.

The usual abbreviation of dioptry is a capital D.; but as this letter is liable to be confused with the symbols of magnitude employed in the formulæ, it seems preferable to follow the usage of Von Rohr and other modern writers on optics who have adopted the abbreviation *dptr.*, although doubtless many will object to this long form.

(optically) in air ($n=1$) at a distance of 1 meter from the vertex, the refracting power of the surface will be 1 dptr. and the radius of the surface will be equal to $(n'-1)$ meters, where n' denotes the index of refraction of the second medium. If the radius of curvature of a mirror is 50 cm., its reflecting power will be 4 dptr. if the reflecting surface is in contact with air ($n=1$), but it will be $5\frac{1}{3}$ dptr. if the surface is in contact with water ($n=\frac{4}{3}$). These examples are given merely to illustrate how the term dioptry is used.

108. Lens-Gauge—The dial of the opticians' lens-gauge described in § 99 is usually graduated so as to give in dioptries the refracting power of the surface which is measured. The refracting power of a spherical refracting surface is proportional to its curvature, as we have seen (§ 105), but it is dependent also on the indices of refraction of the two media. If the first medium is air and if the index of refraction of the second medium is denoted by n, then $F=(n-1)R$. The gauge actually measures the curvature R, and the readings on the dial correspond to the values of R multiplied by the factor $(n-1)$. Direct readings of the refracting power (F) imply, therefore, that the maker has assumed a certain value of the index of refraction n; and if the actual value of n is different from this assumed value, the readings will be erroneous. The value of n assumed by the maker is a constant of the instrument, which should be marked on it, although it may easily be determined empirically by comparing the readings with the determination of the curvature as obtained with an ordinary spherometer.

Suppose that this constant is denoted by c, and that we wish to use the gauge to measure the refracting power (F) of a lens of negligible thickness made of glass of index n. If the refracting powers of the two surfaces of the lens are denoted by F_1 and F_2 and the curvatures by R_1 and R_2, then $F=F_1+F_2$ where $F_1=(n-1)R_1$, $F_2=-(n-1)R_2$, the minus sign in front of the last expression being necessary because the refraction in this case takes place from glass

to air. But if the constant c has a value different from n, the readings of the instrument for the two faces of the lens will not give the correct values F_1, F_2 of the refracting powers. Suppose the readings are denoted by F_1', F_2', so that $F_1' = (c-1)R_1$, $F_2' = -(c-1)R_2$. Then evidently

$$F_1 = \frac{n-1}{c-1} F_1', \qquad F_2 = \frac{n-1}{c-1} F_2',$$

and hence

$$F = \frac{n-1}{c-1} (F_1' + F_2').$$

The gauge-readings must be multiplied therefore by the factor

$$\frac{n-1}{c-1}$$

in order to obtain the correct values of the refracting powers. Suppose, for example, that the graduations on the dial correspond to a value $c = 1.54$ and that the index of the lens to be measured is $n = 1.52$. Then the value of the factor is 0.963; so that if the lens-gauge gives for the refracting power F the value 6.25 dptr., the correct value is obtained by multiplying this value by 0.963, that is, the correct value will be 6.02 dptr.

109. Refraction of Paraxial Rays through a Thin Lens-System.—Let M_1' designate the position of a point conjugate to an axial object-point with respect to an infinitely thin lens of refracting power F_1, and let the point where the axis crosses the lens be designated by A_1. If the lens is surrounded by air, and if we put $u_1 = A_1M_1$, $u_1' = A_1M_1'$, $U_1 = 1/u_1$, $U_1' = 1/u_1'$,
then

$$U_1' = U_1 + F_1.$$

If now at a point A_2 on the axis of the lens beyond A_1 (such that the distance $d = A_1A_2$ is measured in the direction in which the light is going) another infinitely thin lens is set up with its axis in the same straight line with that of the first lens, then M_1' may be regarded as an axial object-point

M_2 with respect to the second lens; and if M_2' designates the position of the point conjugate to M_2 (or M_1') with respect to this lens, then also (supposing that the second lens is surrounded by air and that its refracting power is denoted by F_2),

$$U_2' = U_2 + F_2,$$

where $U_2 = 1/u_2$, $U_2' = 1/u_2'$, $u_2 = A_2M_2 = A_2M_1'$, $u'_2 = A_2M_2'$. Obviously, the point M_2' is the image-point conjugate to the axial object-point M_1 with respect to the two lenses; so that regarding the system as a whole, we may write M, M' in place of M_1, M_2' and U, U' in place of U_1, U_2', respectively.

Now let us impose the condition that *the two thin lenses are in contact with each other* or that they are as close together as possible; in other words, that the axial distance d between the lenses is negligible. If this is the case, the points A_1, A_2 are to be regarded as a pair of coincident points, and hence

$$U_1' = U_2;$$

and, therefore, we may write now:

$$U_1' = U + F_1, \qquad U' = U_1' + F_2.$$

Eliminating U_1', we obtain:

$$U' = U + (F_1 + F_2);$$

and if we put

$$F = F_1 + F_2,$$

we have finally:

$$U' = U + F.$$

Since this formula is seen to be identical in both form and meaning with the formula for a single thin lens, it appears therefore that *a combination of two thin coaxial lenses in contact is equivalent to a single lens of refracting power F equal to the algebraic sum of the refracting powers F_1 and F_2 of the component lenses.*

Theoretically, this rule can be applied to a centered system of *any number of thin lenses in contact.* Thus, the total refracting power of a *thin lens-system* will be

$$F = F_1 + F_2 + \ldots + F_m,$$

where the total number of lenses is denoted by m. This formula may be written:

$$F = \sum_{i=1}^{i=m} F_i,$$

where F_i denotes the refracting power of the ith lens.

In the case of actual lenses placed together in this fashion it will always be a question, How far are we justified in neglecting the total thickness of the system? Two adjacent lenses may be placed in actual contact, but a third lens cannot be in contact with the first. Moreover, even when there are only two lenses, their outward forms may be such that it will not be possible to place them in tangential contact at their vertices, although they can always be made to touch at two points symmetrically situated with respect to their common axis. Attention is directed to this question chiefly in connection with the method of *neutralization* of lenses which is practiced extensively in the fitting of spectacle glasses. Two infinitely thin lenses of equal and opposite refracting powers are said to "neutralize" each other, because when they are placed in contact their total refracting power (F_1+F_2) is equal to zero. Strictly speaking, the neutralization of a negative glass by a positive glass implies not only that the focal lengths are equal in magnitude but also that the primary focal point of one lens shall coincide with the secondary focal point of the other. Both of these conditions are realized in a combination of a plano-concave with a plano-convex lens fitted together so as to form a slab with plane parallel sides. But even with the relatively thin lenses employed in spectacles sensible errors may be introduced by assuming, as is usually done, that the condition $F_1+F_2=0$ is the sole or even the main consideration for neutralization.

110. Prismatic Power of a Thin Lens.—Only such rays as go through the optical center (§ 88) emerge from a lens without being deviated from their original directions. The prismatic power of a thin lens, which, like the power of a

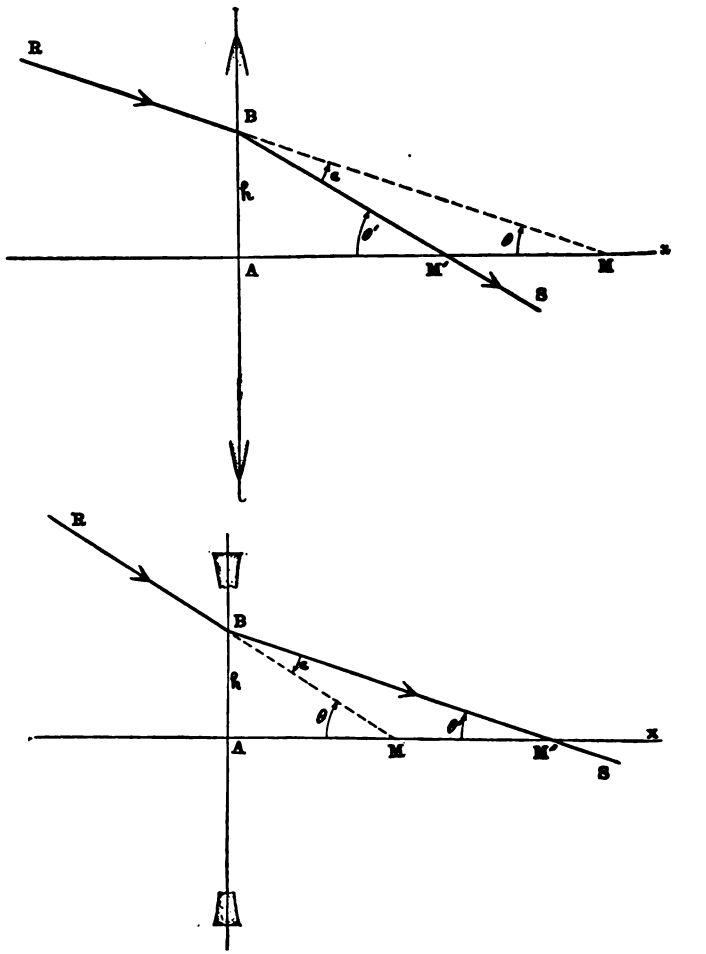


FIG. 141, *a* and *b*.—Prismatic power of infinitely thin lens. (*a*) Convex, (*b*) Concave lens.

thin prism (§ 70), is measured by the deviation of a ray in passing through it, depends not only on the refracting power of the lens but also on the place where the ray enters the lens. In the accompanying diagram (Fig. 141, *a* and *b*)

the point A designates the axial point of a thin lens of refracting power F. A ray RB incident on the lens at B passes out in the direction BS. If M, M′ designate the points where the incident and emergent rays cross the axis, then $\angle M'BM = \epsilon$ is the angle of deviation; and if $\theta = \angle AMB$ denotes the slope of the incident ray and $\theta' = \angle AM'B$ denotes the slope of the emergent ray, evidently we have the relation:

$$\epsilon = \theta - \theta'.$$

The distance $h = AB$ of the incidence-point B from the axis of the lens or the incidence-height of the ray is called by the spectacle-makers the *decentration* of the lens. Since the decentration of an ophthalmic lens is always comparatively small, the ray RB may be regarded as a paraxial ray, and hence we can put θ and θ' in place of $\tan\theta$ and $\tan\theta'$ and write:

$$\theta = -\frac{h}{u} = -h.U, \qquad \theta' = -\frac{h}{u'} = -h.U',$$

where $u = AM$, $u' = AM'$, $U = 1/u$, $U' = 1/u'$, since the lens is supposed to be surrounded by air ($n = 1$). Accordingly,

$$\epsilon = h(U' - U),$$

the deviation-angle ϵ being expressed in radians if h, u and u' are all expressed in terms of the same linear unit. But

$$U' - U = F;$$

and hence

$$\epsilon = h.F \text{ radians.}$$

In this formula the decentration h must be expressed in meters if the refracting power F is given in dioptries. The above relation may be derived immediately also from Fig. 142, where the incident ray RB is drawn parallel to the axis of the lens, so that in this case $\theta' + \epsilon = 0$; and since $\tan\theta' = \theta' = \frac{AB}{F'A} = \frac{h}{f'} = -\frac{h}{f} = -h.F$, we obtain, as above, $\epsilon = h.F$. If a screen is placed perpendicular to the incident light coming in the direction RB, a spot of light will be produced on the screen at the point N where the straight line

RB meets the screen; and if now a lens is interposed at a certain known distance from the screen, the deviation ϵ can easily be determined by measuring the distance NL through which the spot of light is deflected.

However, both the radian and the meter are inconveniently large units for expressing the values of the small mag-

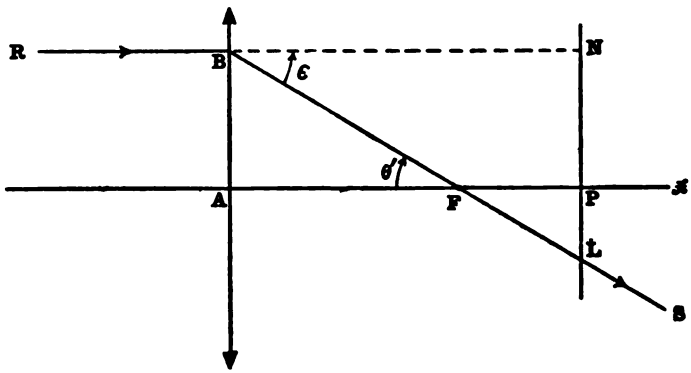


FIG. 142.—Prismatic power of infinitely thin lens; incident ray parallel to axis.

nitudes denoted by ϵ and h. Opticians measure the deviation in terms of the *centrad* or in terms of the *prism-dioptry*, which in the case of small angles, as we have seen (§ 70), is practically the same unit as the centrad. If the angle of deviation expressed in centrads or prism-dioptries is denoted by p, while ϵ denotes the value of this angle in radians, then

$$p = 100\ \epsilon.$$

Moreover, if the decentration h is given in centimeters instead of in meters, we obtain the following formula:

$$p = h.F;$$

that is, *the deviation* (p) *in prism-dioptries* (*or centrads*) *produced by a thin lens in any zone is equal to the product of the refracting power* (F) *of the lens in dioptries by the radius* (h) *of the zone in centimeters;* or as the opticians usually express it, *the prismatic power of a thin lens in prism-dioptries is*

equal to the product of the refracting power of the lens in dioptries by the decentration in centimeters. For example, a spectacle glass of refracting power 5 dptr. must be decentered about 0.4 cm. or 4 mm. in order to have a prismatic power of 2 prism-dptr.

If in Fig. 142 the distance AP of the screen from the lens is 1 meter, the deflection LN in centimeters of the spot of light will be equal to the prismatic power of a lens of focal length $f = AF'$ decentered by the amount $h = AB$.

PROBLEMS

1. How is the curvature of a wave affected by reflection at a plane mirror? How is the curvature of a plane wave affected by reflection at a spherical mirror?

2. The distance between a luminous point and the eye of an observer is 50 cm. A plate of glass ($n = 1.5$), 10 cm. thick, is interposed midway between the point and the eye with its two parallel faces perpendicular to the line of vision. Spherical waves spreading out from the luminous point are refracted through the plate and into the eye. Find the curvature of the wave-front: (*a*) just before it enters the glass, (*b*) immediately after entering the glass, (*c*) immediately after leaving the glass, and (*d*) when it reaches the eye.

Ans. (*a*) -5 dptr.; (*b*) $-3\frac{1}{3}$ dptr.; (*c*) $-3\frac{3}{4}$ dptr.; (*d*) $-2\frac{1}{7}$ dptr.

3. What is the refracting power of a spherical refracting surface of radius 20 cm. separating air ($n = 1$) from glass ($n' = 1.5$)?

Ans. $+2.5$ dptr. or -2.5 dptr., according as the surface is convex or concave, respectively.

4. If the cornea of the eye is regarded as a single spherical refracting surface of radius 7.7 mm. separating air ($n = 1$) from the aqueous humor ($n' = 1.336$), what is its refracting power? Ans. 43.6 dptr.

5. Using the data of the preceding problem, find the refracting power of the cornea when the eye is under water ($n=1.33$). Ans. Nearly 0.78 dptr.

6. What is the reflecting power of a concave mirror of radius 20 cm. when the reflecting surface is in contact with (*a*) air ($n=1$) and (*b*) water ($n=\frac{4}{3}$)?

Ans. (*a*) 10 dptr.; (*b*) 13.33 dptr.

7. A convex spherical surface of radius 25 cm. separates air ($n=1$) from glass ($n'=1.5$). Find the refracting power and the reflecting power of the surface.

Ans. Refracting power is +2 dptr.; reflecting power is −8 dptr.

8. The reflecting power of a spherical mirror in contact with air is +2 dptr. Determine the form of the mirror.

Ans. A concave mirror of radius 1 meter.

9. A spherical mirror is in contact with a liquid of refractive index n. If the reflecting power of the mirror is +2 dptr., show that the mirror is a concave mirror of radius n meters.

10. The index of refraction of carbon bisulphide is 1.629. What is the reflecting power of a concave mirror of radius 25 cm. in contact with this liquid? Ans. +13.032 dptr.

11. What is the refracting power of a thin symmetric convex lens made of glass of index 1.5, if the radius of curvature of each surface is 5 cm.? Ans. +20 dptr.

12. The refracting power of a thin plano-convex lens made of glass of index 1.5 is 20 dptr. Find the radius of the curved surface. Ans. 2.5 cm. or nearly 1 inch.

13. A thin convex meniscus lens is made of glass of index 1.5. The radius of the first surface is 10 and that of the second surface is 25 cm. Assuming that the lens is surrounded by air ($n=1$), find its refracting power.

Ans. +3 dptr.

14. If the lens in the preceding example were made of water of index $\frac{4}{3}$, what will be its refracting power?

Ans. +2 dptr.

15. If the first surface of the lens in No. 13 were in contact with water ($n_1 = \frac{4}{3}$) and the second surface in contact with air ($n_2 = 1$), what will be the refracting power?

Ans. $-\frac{1}{3}$ dptr.

16. If the first surface of the lens in No. 13 were in contact with air ($n_1 = 1$) and the second surface in contact with water ($n_2 = \frac{4}{3}$), what will be the refracting power?

Ans. $+4\frac{1}{3}$ dptr.

17. In examples 13, 14, 15 and 16 suppose the lens were reversed so that the opposite face was turned to the incident light. What would be the answers to these problems then?

Ans. The same answers would be obtained for Nos. 13 and 14; but the answers for Nos. 15 and 16 would be interchanged.

18. Show that the lateral magnification in a spherical mirror, a spherical refracting surface or an infinitely thin lens is equal to the ratio of the reduced "vergences" U and U'.

19. Describe the spherometer and the lens-gauge and explain their principles.

20. Show how a plane wave is refracted through a thin lens, and derive from a diagram for this case the formula for the refracting power.

21. Show how a plane wave is refracted through a thin prism, and derive the formula for the deviation in terms of the refracting angle of the prism and the relative index of refraction.

22. The refracting power of a thin lens is $+6$ dptr. It is made of glass of index 1.5 and surrounded by air ($n = 1$). If the radius of the first surface is $+10$ cm., what is the radius of the second surface? Ans. $r_2 = -50$ cm.

23. A convex lens produces on a screen 14.4 cm. from the lens an image which is three times as large as the object. Find the refracting power of the lens. Ans. 27.78 dptr.

24. A lens-gauge graduated in dioptries for glass of index 1.5 is used to measure a thin double convex lens made

of glass of index 1.6. The readings on the dial give +4 for both surfaces. Find the refracting power of the lens, assuming that its thickness is negligible. Ans. +9.6 dptr.

25. Modern spectacle glasses are meniscus lenses with the concave surface worn next the eye. If the glass is to give the proper correction, it is very important for it to be adjusted at a certain measured distance from the eye. In determining this distance it is necessary to ascertain the "vertex depth" of the concave surface, that is, the perpendicular distance (t) of the vertex from the plane of the edge or contour of the surface. If the diameter of this contour expressed in millimeters is denoted by $2h$, and if the refracting power of the surface next the eye, expressed in dioptries, is denoted by F_2, and, finally, if the index of refraction of the glass is denoted by n, show that the vertex depth of the surface is approximately:

$$t = -0.0005 \frac{h^2 F_2}{n-1} \text{ millimeters.}$$

26. What is the refracting power of a lens which is equivalent to two thin convex lenses of focal lengths 15 and 30 cm., placed in contact? Ans. 10 dptr.

27. A concave lens of focal length 12 cm. is placed in contact with a convex lens of focal length 7.5 cm. Find the refracting power of the combination. Ans. 5 dptr.

28. The refracting power of a thin concave lens is 5 times that of a thin convex lens in contact with it. If the focal length of the combination is 8 cm., find the refracting power of each of the components. Ans. $-15\frac{5}{8}$ and $+3\frac{1}{8}$ dptr.

29. Two thin lenses, made of glass of indices 1.5 and 1.6, are fitted together with the second surface of the first lens coincident with the first surface of the second lens ($r_3 = r_2$). The radii of the surfaces are all positive and equal to 4, 11 and 6 cm. taken in the order named. Find the refracting power of the combination. Ans. 12.5 dptr.

30. What is the prismatic effect of a lens of power +4 dptr. decentered 0.75 cm.? Ans. 3 prism-dioptries.

31. Two thin convex lenses have each a focal length of 1 inch. Find the position of the second focal point of the combination of these two lenses when they are placed with their axes in the same straight line: (*a*) when they are in contact, (*b*) when they are separated by 1.5 inches, and (*c*) when they are separated by 3 inches. Draw a diagram for each case showing the path of a beam of light coming from a distant axial object-point.

Ans. (*a*) Half an inch beyond the combination; (*b*) between the lenses and 1 inch from second lens; (*c*) 2 inches beyond second lens.

32. A convex lens of focal length 20 cm. and a concave lens of focal length 5 cm. are placed 16 cm. apart. Find the positions of the focal points of the combination.

Ans. One of the focal points is 420 cm. from the convex lens and 436 cm. from the concave lens; and the other focal point is 36 cm. from the convex lens and 20 cm. from the concave lens.

33. How much must a lens of 5 dptr. be decentered in order to produce a deviation of 3° 30′? Ans. 1.22 cm.

34. The radius of a spherical surface is measured by a spherometer and found to be 14.857 cm. Measured by a lens-gauge the reading is 3.5 dptr. What is the index of refraction of the glass for which the readings on the dial of the gauge have been calculated? Ans. 1.52.

35. The radii of each surface of a thin symmetric double convex glass lens is 6 inches. The lens is supported with its lower face in contact with the horizontal surface of still water. Assuming that the sun is in the zenith vertically above the lens, and that its apparent diameter is 30′, find the position and size of the sun's image. (Take the indices of refraction of air, glass and water equal to 1, $\frac{3}{2}$ and $\frac{4}{3}$, respectively.)

Ans. A real image 12 inches below the surface of the water, 0.0785 inch in diameter.

CHAPTER IX

ASTIGMATIC LENSES

111. Curvature and Refracting Power of a Normal Section of a Curved Refracting Surface.—The refracting power (F) of a spherical surface is proportional to the curvature (R) of the surface, that is, $F=(n'-n)R$, where n and n' denote the indices of refraction of the media on opposite sides of the surface (§ 105). A spherical surface has the same curvature in every meridian, and hence also its refracting power is uniform, so that the refracted rays in one meridian plane are brought to the same focus as those in another meridian plane. But the surfaces of a lens are not always spherical (§ 87), and therefore, in order to ascertain what happens when a narrow bundle of rays is incident perpendicularly on a curved reflecting or refracting surface of any form, we must investigate the reflecting or refracting power in different sections of the surface; and this means that we must investigate the curvature of these sections. In general, this is a problem of some difficulty and involves a more or less extensive knowledge of the theory of curved surfaces and the methods of infinitesimal geometry. No attempt can be made to explain this theory here, but for the student who is not already familiar with it, certain general definitions and propositions of geometry which have a direct bearing on the optical problems to be treated in this chapter will be stated as succinctly as possible.

The normal to a curved surface at any point is a straight line drawn perpendicular to the tangent plane at that point. The curved line which is traced on the surface by a plane containing the normal at a point A of the surface is called

a *normal section* through this point. The normal sections of a sphere, like the meridians of longitude of the earth (assumed to be a perfect sphere), are all great circles of the sphere, and their curvatures are equal. But, generally,

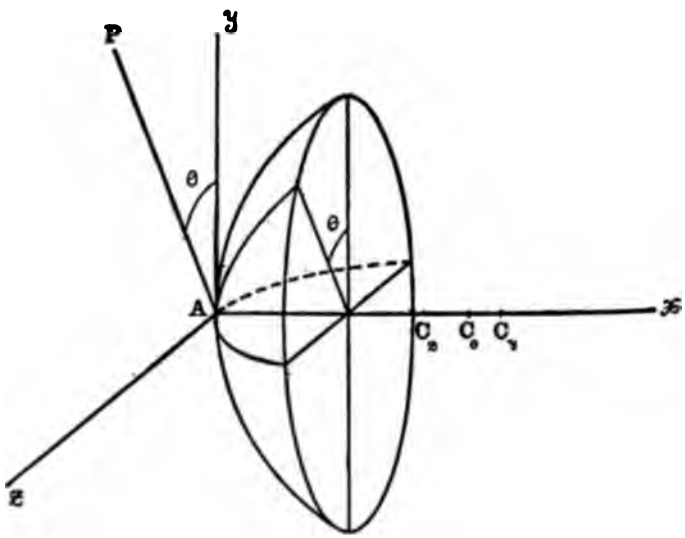


FIG. 143.—Normal sections of curved surface: xAy and xAz planes of principal sections; xAP plane of oblique normal section.

the curvatures of the normal sections through a point on a curved surface will vary from one section to the next; so that if we imagine a plane containing the normal to be turned around this line as axis, we shall find that for one special azimuth of this revolving plane the curved line which it carves out on the surface will have the greatest curvature, and that then as the plane continues to revolve the curvature of the section decreases and reaches its least value for an azimuth which is exactly 90° from that for which the curvature was greatest. Thus, for example, in a cylindrical surface the curvature at any point is least and equal to zero in a normal section whose plane is parallel to the axis of the cylinder, and it is greatest in a normal section made

by a plane perpendicular to the axis. *At each point* A *of a curved surface the normal sections of greatest and least curvatures lie always in two perpendicular planes*, which are called the planes of the *principal sections* of the surface at A. The lines of intersection of these planes with each other and with the tangent plane at A may be chosen as the axes of reference of a system of rectangular coördinates x, y, z whose x-axis is the normal Ax (Fig. 143). The centers of curvature of the principal sections made by the xy-plane and the xz-plane will be designated by C_y and C_z, respectively; and the curvatures of the principal sections will be denoted by R_y and R_z, so that if $r_y = AC_y$ and $r_z = AC_z$ denote the principal radii of curvature of the surface at the point A, we must have here (§ 99) $R_y = 1/r_y$ and $R_z = 1/r_z$.

Now there is a remarkable geometrical relation between the curvature of any normal section at A and the curvatures of the principal sections of the surface at this point which will be stated also without giving the proof. Let a plane containing the normal Ax intersect the tangent plane (or yz-plane) in the straight line AP (Fig. 143) and put $\angle yAP = \theta$. The center of curvature of the normal section made by this plane lies also on the normal Ax at a point which may be designated as C_θ, so that the radius of curvature is $AC_\theta = r_\theta$, and the curvature itself is $R_\theta = 1/r_\theta$. The connection between R_θ and the principal curvatures R_y and R_z is expressed by the following formula:

$$R_\theta = R_y.\cos^2\theta + R_z.\sin^2\theta,$$

where θ denotes the angle which the normal section makes with the xy-plane.

In a normal section at right angles to the first we should have, therefore,

$$R_{\theta+90^\circ} = R_y.\cos^2(\theta+90^\circ) + R_z.\sin^2(\theta+90^\circ),$$

or, since $\cos(\theta+90^\circ) = -\sin\theta$, $\sin(\theta+90^\circ) = \cos\theta$,

$$R_{\theta+90^\circ} = R_y.\sin^2\theta + R_z\cos^2\theta.$$

Adding the curvatures $R\theta$ and $R\theta+90°$, we obtain the relation:

$$R\theta+R\theta+90°=R_y+R_z;$$

that is, *the algebraic sum of the curvatures of any two normal sections intersecting each other at right angles at a point on a curved surface has a constant value, which is equal to the algebraic sum of the principal curvatures at this point.*

These theorems concerning the curvatures of the normal sections at a point of a curved surface are due to the great mathematician EULER (1707–1783), who made notable contributions also to the theory of optics.

Since, therefore, the curvature of a surface at the point A varies from one azimuth to another as has just been explained, the power of a refracting surface will vary in exactly the same way. Accordingly, the principal sections for which the curvature of a refracting surface has its greatest and least values (R_y, R_z) are also the sections at this place of greatest and least refracting powers (F_y,F_z), because

$$F_y=(n'-n)R_y, \qquad F_z=(n'-n)R_z.$$

The refracting power at this place in an oblique normal section which is inclined to the xy-plane at an angle θ will be:

$$F\theta=(n'-n)R\theta;$$

and the relation between $F\theta$ and F_y, F_z is given by the formula:

$$F\theta=F_y.\cos^2\theta+F_z.\sin^2\theta;$$

and moreover:

$$F\theta+F\theta+90°=F_y+F_z;$$

that is, *the algebraic sum of the refracting powers in any two normal sections through a point on a curved refracting surface is constant and equal to the algebraic sum of the principal refracting powers.*

For example, in Fig. 144, let A designate a point of a curved refracting surface, and let the normal at this point be represented by the straight line Ax, which in accordance with the preceding discussion is to be taken as the x-axis of a system of rectangular coördinates with its origin at A.

The y-axis is represented by a straight line drawn in the plane of the paper perpendicular to Ax. The plane of the paper represents the plane of one of the principal sections, whereas the xz-plane at right angles to this plane represents

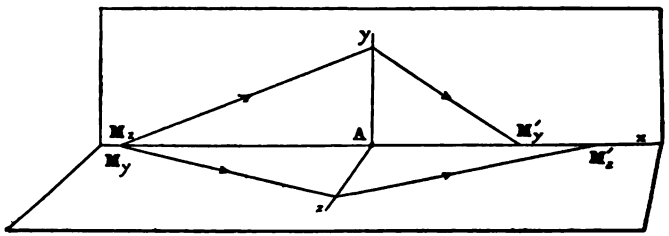


FIG. 144.—Chief ray of narrow bundle of rays normal to curved refracting surface: Principal sections xAy, xAz; tangent plane yAz.

the plane of the other principal section. The tangent-plane at A is represented by the yz-plane perpendicular to the normal. Consider now a narrow bundle of rays which proceeding from a point M on the normal are incident on the curved refracting surface at points which are all very close to A. This point M may be designated also by M_y or by M_z according as it is regarded as lying in the one or the other of the two principal sections; or it may be designated also by M_θ if it is to be considered as lying in an oblique normal section which is inclined to the xy-plane at an angle θ. The chief ray of the bundle is the ray which coincides with the normal to the surface at A and which proceeds therefore into the second medium without being deviated. A plane containing this chief ray will cut out from the bundle a pencil of rays which will be refracted at points of the surface which lie in a normal section. The pencil of rays proceeding from M_y in the xy-plane will be refracted to a point M_y', while the pencil of rays proceeding from M_z will be refracted to a point M_z'; and, in general, these points M_y' and M_z' will be two different points on the normal Ax. Now if U_y, U_y' denote the reduced "vergences" (§ 106) of the pair of conju-

gate points M_y, M_y' in one principal section; and, similarly, if U_z, U_z' denote the reduced "vergences" of the pair of conjugate points M_z, M_z' in the other principal section, evidently we shall have the following relations:

$$U_y' = U_y + F_y, \qquad U_z' = U_z + F_z.$$

Similarly, also, a pencil of rays proceeding from $M\theta$ and meeting the refracting surface at points in an oblique normal section will be refracted to a point $M\theta'$ which will lie on Ax between M_y' and M_z', so that

$$U\theta' = U\theta + F\theta.$$

If the bundle of incident rays is homocentric, that is, if the points designated by M_y, M_z and $M\theta$ are all coincident, then $U_y = U_z = U\theta = U$. The peculiarity of the imagery consists in the fact that instead of obtaining a single image-point M' corresponding to an object-point M, as in the case of a spherical refracting surface, we find here a whole series of such points lying on the segment $M_y'M_z'$ of the normal Ax. This will be explained more fully in § 113.

112. Surfaces of Revolution. Cylindrical and Toric Surfaces.—The curved reflecting and refracting surfaces of optical mirrors and lenses are almost without exception *surfaces of revolution*, that is, surfaces generated by the revolution of the arc of a plane curve around an axis in its plane. Accordingly, it is desirable to call attention to some of the special properties of these surfaces. The curve traced on a surface of revolution by a plane containing the axis of revolution is called a *meridian section*. The normals to the generating curve are also normals to the surface; and since the normal at any point of the surface lies in the meridian section which passes through that point, it follows that the normals to a surface of revolution all intersect the axis of revolution.

The two principal sections at any point of a surface of revolution are the meridian section which passes through that point and the normal section which is perpendicular to the meridian section. The center of curvature of the

latter principal section lies on the axis of revolution at the point where the normal crosses it.

Not only are the surfaces of mirrors and lenses generally surfaces of revolution, but usually they are very simple types of such surfaces. A spherical surface may be considered as generated by the revolution of a circle around one of its diameters. The other chief forms of reflecting and refracting surfaces are *cylindrical* and *toric surfaces*, which are also comparatively easy to grind.

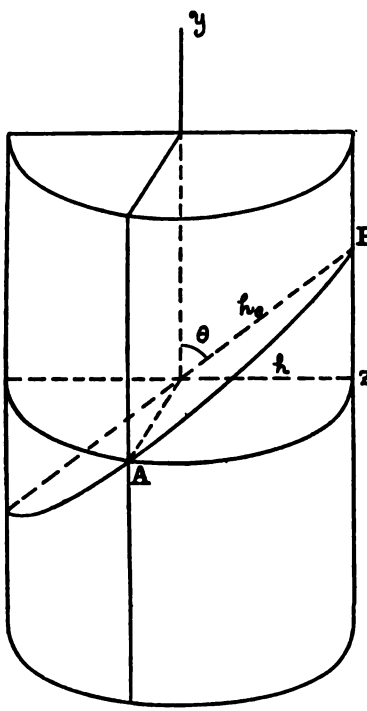


FIG. 145.—Refracting power of cylindrical surface: Principal sections made by planes Ay and Az; oblique section AP.

A *cylindrical surface* of revolution is generated by the revolution of a straight line about a parallel straight line as axis, called the *axis of the cylinder*. A meridian section of a cylinder at a point A on the surface (Fig. 145) will be a straight line of zero curvature, whereas the other principal section at right angles to the axis of the cylinder will be the arc of a circle whose curvature is $R=1/r$, where r denotes the radius of the cylinder. If the y-axis is drawn parallel to the cylinder-axis, then $R_y=0$, $R_z=R$; and hence according to EULER'S formula given in § 111, the curvature in an oblique normal section AP inclined to the axis of the cylinder at an angle θ will be

$$R_\theta = R.\sin^2\theta.$$

This result may be obtained also independently by observing that although the arcs Az and AP in Fig. 145 have the same *sagitta* (§ 99), their chords denoted by $2h$ and $2h_\theta$ are unequal in length, because $h = h_\theta . \sin\theta$. Now the curvatures of two arcs having the same *sagitta* are inversely proportional to the squares of their chords; consequently,

$$\frac{R_\theta}{R} = \frac{h^2}{h_\theta{}^2},$$

and hence

$$R_\theta = R . \sin^2\theta,$$

exactly as above. Moreover, in a normal section perpen-

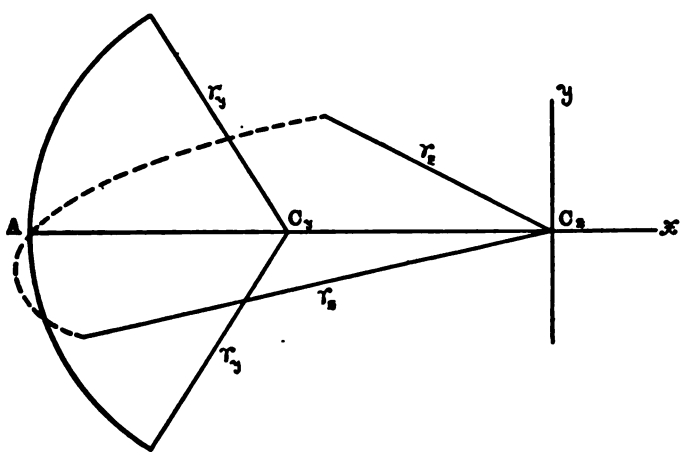


FIG. 146.—Principal sections of toric surface.

dicular to the section AP, we find, by writing $(\theta+90°)$ in place of θ,

$$R_{\theta+90°} = R . \cos^2\theta;$$

and therefore

$$R_\theta + R_{\theta+90°} = R.$$

Accordingly, in the case of a cylindrical refracting surface, if the maximum refracting power is denoted by F, the refracting power in an oblique section inclined to the

axis at an angle θ will be $F.\sin^2\theta$, and in a section at right angles to this $F.\cos^2\theta$. The refracting power F of a cylindrical refracting surface may, therefore, be considered as in a certain sense capable of resolution into a refracting power $F.\sin^2\theta$ in one oblique section and a refracting power

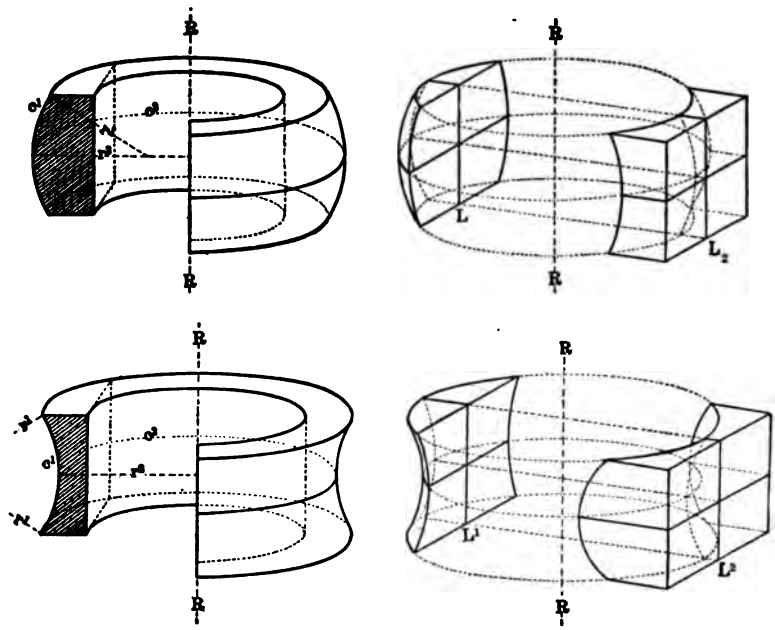


FIG. 147, *a* and *b*.—Toric surfaces (reproduced from PRENTICE's *Ophthalmic Lenses and Prisms* by permission of the author).

$F.\cos^2\theta$ in a section at right angles to the first; and since

$$F\theta + F\theta + 90° = F,$$

we can say that *the algebraic sum of the refracting powers in any two mutually perpendicular sections of a cylindrical refracting surface is constant and equal to the maximum refracting power.*

A *toric* or *toroidal surface* (so-called from the architectural term *torus* applied to the molding at the base of an Ionic column) is a surface shaped like an anchor-ring which

is generated by the revolution of a conic section around an axis which lies in the plane of the generating curve but does not pass through its center. The surface of an automobile tyre is a toric surface, being generated by the revolution of the circular cross-section of the tyre around an axis per-

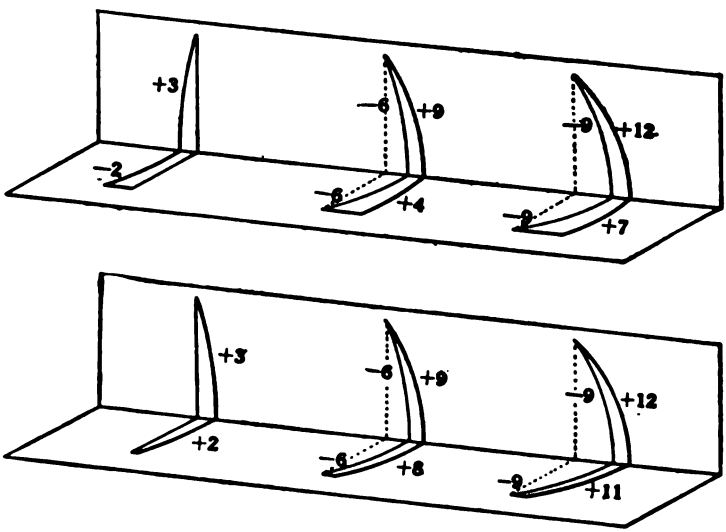


Fig. 148, *a* and *b*.—Principal sections of toric lenses (reproduced from Prentice's *Ophthalmic Lenses and Prisms* by permission of the author).

pendicular to the plane of the wheel at its center. Toric refracting surfaces are generated always by the revolution of the arc of a circle (Fig. 146). The arcs of the two principal sections of a toric surface of a lens bisect each other at the vertex A of the surface, so that the normal Ax is an axis of symmetry. If the axis of revolution is parallel to the y-axis of the system of rectangular coördinates, the center of the meridian section through A is at the center C_y of the generating circle, whereas the center of the other principal section at A is at the point of intersection C_s of the normal Ax with the axis of revolution.

The diagrams, Fig. 147, *a* and *b* (which are copied from

the beautiful drawings of Mr. PRENTICE in his valuable and original essay on "Ophthalmic Lenses and Prisms" in the *American Encyclopædia of Opthhalmology*) show the two principal forms of toric surfaces. The principal sections of some types of toric lenses are indicated in Fig. 148, *a* and *b*.

A cylindrical surface of revolution may be considered as a special form of toric surface by regarding the segment of the generating straight line as the arc of a circle with an infinite radius.

113. Refraction of a Narrow Bundle of Rays incident Normally on a Cylindrical Refracting Surface. Sturm's Conoid.—In order to obtain a clear idea of the character of a bundle of rays refracted at a cylindrical surface or through a thin cylindrical lens, suppose, by way of illustra-

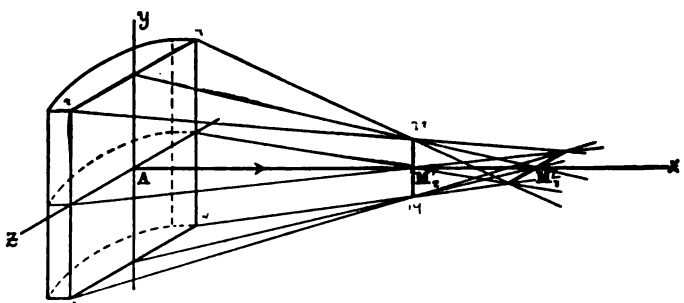


FIG. 149.—Chief ray of narrow bundle meets cylindrical refracting surface normally; astigmatic bundle of refracted rays. Principal sections *xAy* and *xAz*.

tion, that we consider a special case of the problem which we had in § 111 in connection with Fig. 144, namely, the case in which a narrow homocentric bundle of incident rays, originally converging towards a point M, is intercepted before it reaches this point by being received on a cylindrical refracting surface which is placed so that the chief ray of the bundle meets the surface normally at a point A and proceeds, therefore, along the normal A*x* (Fig. 149) without being deflected.

For convenience of delineation, the cylindrical surface is represented in the figure as the first surface of an infinitely thin plano-cylindrical lens, but the explanation is not essentially affected by the fact that it applies to a bundle of rays which have undergone also a second refraction at the plane face of the lens. The bundle of incident rays is not represented in the figure. The point where the chief ray meets the lens is designated by A. In the drawing this point A is marked on the second or plane face of the lens, but since the lens is supposed to be infinitely thin, this point may be regarded also as lying on the first face. The plane of the paper represents the meridian section of the cylindrical surface through the vertex A, and hence the axis of the cylinder is in this plane and parallel to the straight line Ay perpendicular to Ax in the meridian or xy-plane. This meridian plane is one of the principal sections at the vertex A of the cylindrical surface; whereas the other principal section is the xz-plane at right angles to the plane of the paper. The bundle of rays is cut by these principal sections in a *pencil of meridian rays* lying in the meridian xy-plane and a *pencil of sagittal rays* (named by analogy with the so-called "sagittal suture" in anatomy) lying in the xz-plane; the chief ray of the bundle being common to both of these pencils, since it is the line of intersection of the two principal sections of the bundle. Now the meridian rays traversing the infinitely thin cylindrical lens in a section containing the axis of the cylinder will be entirely unaffected in transit and will proceed therefore to the point M just as though the thin piece of glass had not been interposed in the way; so that this point regarded now as the point of rendezvous, so to speak, of the meridian rays after they have passed through the lens may also be designated by M_y', as in fact it is marked in the diagram. On the other hand, the rays of the sagittal pencil meet the surface in points lying on the arc of the section made by the xz-plane, and the rays in this plane are refracted just as they would be through a plano-spherical

lens of the same curvature as that of the cylinder; and accordingly after passing through the lens they will be brought to a focus at a point M_z' on the chief ray Ax, which in the case here supposed will be between the lens and the point M_y', as represented in the figure.

The bundle of rays after refraction is no longer homocentric, so that an object-point is not reproduced in a cylindrical lens by a single image-point or even by a pair of image-points, since only the meridian and sagittal image-rays intersect in the so-called image-points M_y' and M_z', respectively. Under such circumstances, the bundle of image-rays is said to be *astigmatic* (or without focus), which, in fact, is the general character of a bundle of optical rays, as will be further explained in Chapter XV.

Rays which are incident on the cylindrical surface in an oblique section made by a plane containing the normal Ax will be brought to a focus at a point lying between M_y' and M_z', as explained in § 111. But the two points M_y' and M_z' have a superior right to be regarded as the image-points of the astigmatic bundle of rays, not only because they are the image-points of the two principal pencils of the bundle, but also because the so-called *image-lines* of the astigmatic bundle of rays are located at these places, as we shall proceed to show.

Imagine a straight line drawn on the surface of the cylinder parallel to the y-axis and at a short distance from the xy-plane, and consider the pencil of rays which meet the surface in points lying along this line; these rays after passing through the lens will meet in a point in the xz-plane a little to one side of the image-point M_y'; and the assemblage of these image-points will form a very short image-line perpendicular to the meridian section of the bundle of rays at the point M_y'; just as though the pencil of meridian rays had been rotated through a very small angle around an axis parallel to the y-axis and passing through M_z'. And, similarly, if the pencil of sagittal rays is rotated slightly

on both sides of the xz-plane around an axis parallel to the z-axis and passing through the image-point M_y', the image-point M_z' will trace out a little image-line perpendicular to the sagittal section of the astigmatic bundle of rays. Thus, instead of a point-like image of a point-like object or point-to-point correspondence between object and image, that is, instead of the so-called *punctual imagery* which we have when paraxial rays are reflected or refracted at a spherical surface, we obtain here something essentially different; for in this case each point of the object is reproduced by *two*

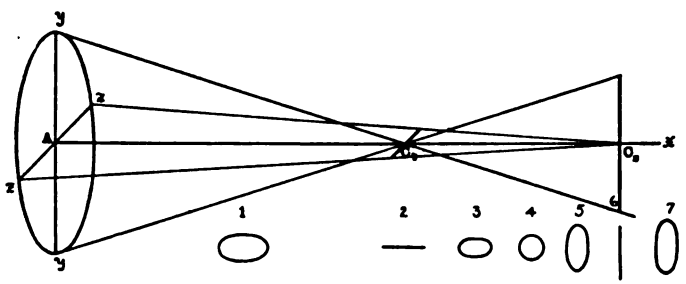


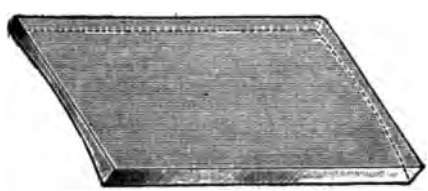
FIG. 150.—STURM'S conoid.

tiny image-lines, each perpendicular to the chief ray of the bundle, one in one principal section and the other in the other principal section; so that if one of the image-lines is vertical, the other will be horizontal. The image-line which passes through the image-point of the meridian rays lies in the plane of the sagittal section, and *vice-versa.*

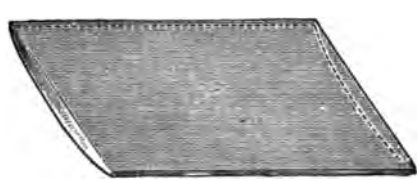
The case in which an object-point is reproduced by two short image-lines is the simplest form of astigmatism, and it is only under exceptionally favorable circumstances that it can be actually realized as described above. The astigmatic bundle of rays represented in Fig. 150, which is completely symmetrical in the two principal sections is known as STURM'S *conoid* after the celebrated mathematician who appears to have been the first to make a systematic investigation (1838) of the characteristics of a narrow bundle of

optical rays. If the lens-opening is determined by a small circular stop in a plane at right angles to the optical axis (or x-axis) and with its center on this axis, the transverse sections of the astigmatic bundle of refracted rays made by planes perpendicular to the chief ray (that is, parallel to the yz-plane) will be ellipses with their major axes parallel to the y-axis in one part of the bundle and parallel to the z-axis in the other part. These elliptical sections become narrower and narrower as they approach either of the image-lines, at both of which places the elliptical section collapses into the major-axis of the ellipse. At some intermediate point between the two image-lines the section of the bundle will be a circle (the so-called "circle of least confusion").

114. Thin Cylindrical and Toric Lenses.—Optical lenses may now be classified in two principal groups, namely, *anastigmatic* (or simply *stigmatic*) *lenses* and *astigmatic lenses*, according as the imagery produced by the refraction of par-



a, Concave.



b, Convex.

FIG. 151, *a* and *b*.—Plano-cylindrical lenses.

axial rays through the lens is punctual imagery or not (§ 113). Anastigmatic lenses are single focus lenses, whereas astigmatic lenses may be said to be double focus lenses. The essential requirement is that the optical axis of the lens,

which is generally an axis of symmetry, shall meet both faces normally (§ 87); and another condition that must always be fulfilled in an actual lens is that the planes of the principal sections at the vertex of the first surface shall also be the planes of the principal sections at the vertex of the second surface. Astigmatic lenses are generally cylindrical or toric.

Cylindrical lenses are made in three forms, namely, *plano-cylindrical* (one surface cylindrical and the other plane,

FIG. 152.—Sphero-cylindrical lens.

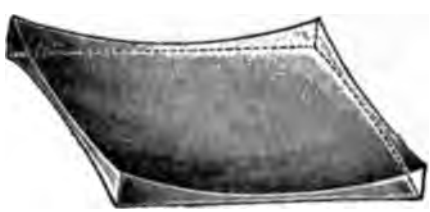


FIG. 153.—Sphero-cylindrical lens.

Fig. 151, *a* and *b*), *cross-cylindrical* (both surfaces cylindrical, the axes of the cylinders being at right angles), and *sphero-cylindrical* (one surface cylindrical and the other spherical, Figs. 152 and 153). All of these forms are quite common in modern spectacle glasses, but prior to 1860 cylindrical lenses were hardly employed at all. The first scientific use of a cylindrical lens seems to have been made by FRESNEL (1788–1827) in 1819 for the purpose of obtaining a luminous line. In 1825 Sir GEORGE AIRY (1801–1892), afterwards the distinguished astronomer-royal at Greenwich, employed a con-

cave sphero-cylindrical glass to correct the myopic astigmatism of one of his eyes. But it was not until DONDERS (1818–1889) published his treatise on astigmatism and cylindrical glasses in 1862 that their importance began to be recognized by ophthalmologists all over the world.

In a toric lens usually only one of the surfaces is toric (§ 112), while the other is plane or spherical. The diagrams, Fig. 147, *a* and *b*, and Fig. 148 show the principal types of toric lenses.

Let $F_{y,1}$, $F_{y,2}$ and $F_{z,1}$, $F_{z,2}$, denote the refracting powers of the two surfaces of an astigmatic lens in the xy-plane and xz-plane, respectively, which are the planes of the principal sections of the thin lens with respect to its optical center A. Now the total refracting power (F) of a thin lens was found (§ 105) to be equal to the algebraic sum (F_1+F_2) of the powers of the two surfaces of the lens; so that applying this formula to an astigmatic lens, we obtain for the refracting power in the two principal sections:

$$F_y=F_{y,1}+F_{y,2}, \qquad F_z=F_{z,1}+F_{z,2}.$$

In each of the following special cases the lens is supposed to be surrounded by the same medium (n) on both sides, while the index of refraction of the lens itself is denoted by n'.

(1) Consider, first, the case of a *plano-cylindrical lens*, which in a principal section containing the axis of the cylinder acts, as was remarked (§ 113), like a slab of the same material with plane parallel faces; whereas in the other principal section the effect is the same as that of a plano-spherical lens of the same radius (r) as that of the cylinder. If the axis of the cylinder is parallel to the y-axis, and if the plane surface is supposed to be the second surface, we shall have in this case:

$$F_{y,1}=F_{y,2}=F_{z,2}=0,$$

and, consequently:

$$F_y=0, \qquad F_z=F_{z,1}=F=(n'-n)R,$$

where F denotes the maximum refracting power of the cylindrical surface, and $R=1/r$ denotes its curvature.

If M designates the position of an object-point lying on the optical axis (x-axis) of a thin plano-cylindrical lens, and if $M\theta'$ designates the position of the corresponding image-point produced by the refraction through the lens of the rays which lie in the plane of a normal section inclined at an angle θ to the axis of the cylinder; and if we put

$$AM = u, \qquad AM\theta' = u', \qquad U = n/u, \qquad U\theta' = n/u\theta',$$

then

$$U\theta' = U + F\theta, \text{ where } F\theta = F.\sin^2\theta;$$

and for the two principal sections:

$$U_y' = U, \qquad U_z' = U + F.$$

(2) In a *cross-cylindrical lens* the axes of y and z are parallel to the axes of the cylinders. Assuming that the cylindrical axis of the first surface of the lens is parallel to the y-axis, we have for a thin lens of this form:

$$F_{y,1} = F_{z,2} = 0,$$

$$F_y = F_{y,2} = -(n'-n)R_2, \qquad F_z = F_{z,1} = (n'-n)R_1,$$

$$F = (n'-n)(R_1.\sin^2\theta - R_2.\cos^2\theta);$$

where R_1, R_2 denote the maximum curvatures of the cylinders and $F\theta$ denotes the refracting power in a section inclined at an angle θ to the axis of the first surface.

(3) In a thin *sphero-cylindrical lens*, if we suppose, for example, that the axis of the cylindrical surface is parallel to the y-axis and that this surface is also the first surface of the lens, then

$$F_{y,1} = 0, \qquad F_{y,2} = F_{z,2} = F_2,$$

$$F_y = F_{y,2} = -(n'-n)R_2,$$

$$F_z = F_{z,1} + F_y = (n'-n)(R_1 - R_2),$$

$$F\theta = (n'-n)(R_1.\sin^2\theta - R_2);$$

where R_1, R_2 denote the maximum curvatures of the cylindrical and spherical faces, respectively, and $F\theta$ denotes the refracting power of the combination in a plane inclined at an angle θ to the axis of the cylinder.

(4) Consider, finally, a thin *toric lens*, whose second face may be supposed to be spherical, so that if r_2 denotes the radius of this surface, its refracting power will be

$F_2 = -(n'-n)R_2$, where $R_2 = 1/r_2$. Then if $R_{y,1}$, $R_{x,1}$ denote the principal curvatures of the toric surface, the refracting powers of the lens will be

$$F_y = (n'-n)\ (R_{y,1}-R_2), \qquad F_x = (n'-n)\ (R_{x,1}-R_2),$$
$$F_\theta = (n'-n)\ (R_{y,1}.\cos^2\theta + R_{x,1}.\sin^2\theta - R_2).$$

115. Transposing of Cylindrical Lenses.—The orientation of a cylindrical refracting surface is described by assigning the value of the angle φ which the axis of the cylinder makes with a fixed line of reference. In a cylindrical spectacle glass this line of reference is a horizontal line usually imagined as drawn from a point opposite the center of the patient's eye either towards his temple or towards his nose;

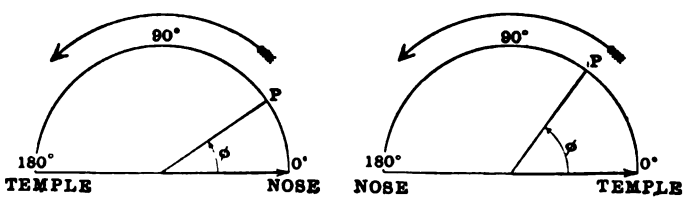


FIG. 154.—Mode of reckoning axis of cylindrical eye-glass.

and the angle through which this line has to be rotated in a vertical plane in order for it to be parallel to the axis of the cylinder is the angle denoted by φ. In England and America it is customary to imagine the horizontal line of reference as drawn *from the center of the glass towards that temple of the patient which is on the right-hand side of an observer* supposed to be adjusting the glass on the patient's eye; so that for a glass in front of either eye the radius vector is supposed to rotate in a counter-clockwise sense from 0° to 180°, as represented in Fig. 154. A different plan was recommended by the international ophthalmological congress which met in Naples in 1909, whereby the angle φ was to be reckoned from an initial position of the radius vector drawn horizontally from a point opposite the center of the eye *towards the nose*. According to this plan, the sense of

rotation will be clockwise for one eye and counter-clockwise for the other eye, as represented in Fig. 155.

A sphero-cylindrical glass is described in an ophthalmological prescription by giving the refracting power P of the cylindrical component and the refracting power Q of the spherical component, together with the slope φ of the axis of the cylinder, in a formula which is usually written as follows:

$$Q \text{ sph.} \frown P \text{ cyl., ax.} \varphi,$$

where the symbol $\frown$ means "combined with."

Opticians speak of *transposing* a lens when they substitute a glass of one form for an equivalent glass of another

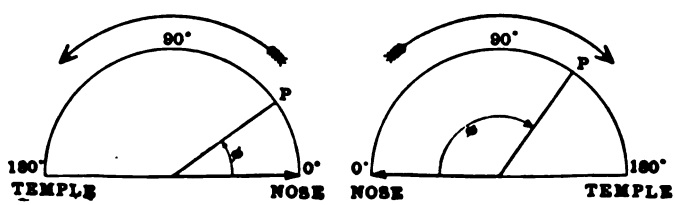


FIG. 155.—Mode of reckoning axis of cylindrical eye-glass.

form. All that is necessary for this purpose is to see that the powers of the lens in the two principal sections remain the same as before. The following *rules for transposing cylindrical lenses* may be useful:

(1) *To transpose a sphero-cylindrical lens into another sphero-cylindrical lens or into a cross-cylindrical lens:*

A lens given by the formula Q sph. $\frown$ P cyl., ax. φ is equivalent to either of the following combinations:

a. Sphero-cylinder: $(P+Q)$ sph. $\frown$ $-P$ cyl., ax. $(\varphi \pm 90°)$

b. Cross-cylinder: $(P+Q)$ cyl., ax. $\varphi \frown Q$ cyl., ax. $(\varphi \pm 90°)$.

The power of the spherical component in the original combination is Q dptr. in both principal sections, and the power of the cylindrical component is P dptr. in the section which is inclined to the line of reference at an angle $(\varphi \pm 90°)$; so that the combined power in this latter section is $(P+Q)$ dptr.

Accordingly, a spherical surface of power $(P+Q)$ dptr. must be combined with a cylindrical surface of power $-P$ dptr. and of axis-slope $(\varphi \pm 90°)$. With respect to the double sign in the expression $(\varphi \pm 90°)$, the rule is to select always that one of the two signs which will make the slope of the cylinder-axis positive and less than 180°. Thus, for example, +8 dptr. sph. $\bigcirc$ +2 dptr. cyl., ax. 20° is equivalent to +10 dptr. sph. $\bigcirc$ −2 dptr. cyl., ax. 110° or to +10 dptr. cyl., ax. 20° $\bigcirc$ +8 dptr. cyl., ax. 110°.

(2) *To transpose a cross-cylindrical lens into a sphero-cylindrical lens:*

The combination P cyl., ax. φ $\bigcirc$ R cyl., ax. $(\varphi \pm 90°)$ is equivalent to either of the following:

a. Sphero-cylinder: P sph. $\bigcirc$ $(R-P)$ cyl., ax. $(\varphi \pm 90°)$, or

b. Sphero-cylinder: R sph. $\bigcirc$ $(P-R)$ cyl., ax. φ.

Thus, +2 cyl., ax. 80° $\bigcirc$ +3 cyl., ax. 170° may be replaced by either +2 sph. $\bigcirc$ +1 cyl., ax. 170° or +3 sph. $\bigcirc$ −1 cyl., ax. 80°.

(3) *To transpose a spherical lens into a cross-cylinder:*

Q sph. is equivalent to Q cyl., ax. φ $\bigcirc$ Q cyl., ax. $(\varphi \pm 90°)$, where the angle φ may have any value between 0° and 180°. For example, +5 sph. is equivalent to +5 cyl., ax. 10° $\bigcirc$ +5 cyl., ax. 100°.

(4) The refracting powers of a *toric* surface in the principal sections are $F_y=(n'-n)/r_y$ and $F_z=(n'-n)/r_z$. Let us suppose that the axis of revolution is parallel to the y-axis. The toric refracting surface may be replaced by a sphero-cylindrical lens in either of two ways, as follows:

a. F_z sph. $\bigcirc$ (F_y-F_z) cyl., axis parallel to y-axis.

b. F_y sph. $\bigcirc$ (F_z-F_y) cyl., axis parallel to z-axis.

116. Obliquely Crossed Cylinders.—Oculists and optometrists sometimes prescribe a bi-cylindrical spectacle-glass with the axes of the cylinders crossed, not at right angles (as in the so-called cross-cylinder), but at an acute or obtuse angle γ; and as it is not easy to grind a lens of this form, the optician prefers to make an equivalent sphero-cylinder

or a cross-cylinder, which will have precisely the same optical effect as the prescribed combination of obliquely crossed cylinders. His problem may be stated thus:

Being given the refracting powers F_1, F_2 of the two surfaces of the bi-cylindrical lens, and the angle γ between the directions of the axes of the cylinders, it is required to calculate the refracting powers P and Q of the cylindrical and spherical components, respectively, of the equivalent sphero-cylindrical combination, together with the direction of the axis of the cylinder; that is, it is required to transpose

$$F_1 \text{ cyl., ax. } \varphi \subset F_2 \text{ cyl., ax. } (\varphi+\gamma)$$

into

$$Q \text{ sph.} \subset P \text{ cyl., ax. } (\varphi+\alpha).$$

Simple working formulæ for converting one of these lenses into the other were developed first by Mr. CHARLES F. PRENTICE. The following method is based on an article "On obliquely crossed cylinders" by Professor S. P. THOMPSON published in the *Philosophical Magazine* (series 5, xlix., 1900, pp. 316–324).

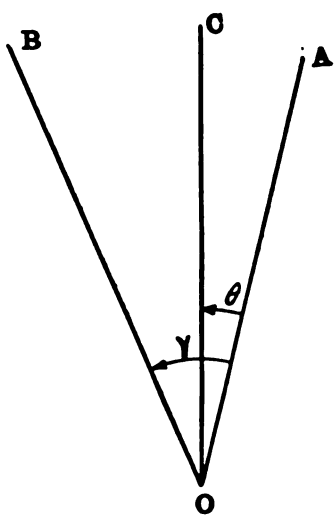


FIG. 156.—Axes of obliquely crossed cylinders.

In Fig. 156 the straight lines OA and OB are drawn parallel to the cylindrical axes of the bi-cylindrical lens, so that $\angle AOB = \gamma$. Through O draw another straight line OC, and let $\angle AOC$ be denoted by θ. In the section of the lens at right angles to OC the total refracting power will be (see § 112):

$$F_1.\cos^2\theta + F_2.\cos^2(\gamma-\theta);$$

and in the section containing OC:

$$F_1.\sin^2\theta+F_2.\sin^2(\gamma-\theta).$$

The sum of these two expressions is equal to (F_1+F_2); and according to the theory of curved surfaces (§ 111), this sum must also be equal to the sum of the maximum and minimum refracting powers of the equivalent sphero-cylindrical lens. Now, obviously, $(P+Q)$ will be the maximum (or minimum) refracting power in a section of the latter lens at right angles to the axis of the cylinder, whereas Q will be the minimum (or maximum) refracting power in the section containing the axis of the cylinder; accordingly, first of all, we find that we must have:

$$2Q+P=F_1+F_2.$$

Now there is a certain value of the angle θ, say, $\theta=\alpha$, for which the first of the two expressions above will be a maximum (or minimum) and the second a minimum (or maximum); and if we can determine this angle α, the problem will practically be solved, because then we shall have:

$$P+Q=F_1.\cos^2\alpha+F_2.\cos^2(\gamma-\alpha),$$
$$Q=F_1.\sin^2\alpha+F_2.\sin^2(\gamma-\alpha);$$

where (on the assumption that Q is the *minimum* refracting power in the section containing the axis of the cylinder) α denotes *the angle between the cylindrical axis of the sphero-cylinder and the cylindrical axis of the cylinder whose refracting power is denoted by* F_1. Now in order to ascertain this angle α, all we have to do (as will be obvious to any one who is familiar with the elements of the differential calculus) is, first, to differentiate the expression

$$F_1.\cos^2\theta+F_2.\cos^2(\gamma-\theta)$$

with respect to θ, and then, after writing α in place of θ, to put the resultant expression equal to zero. Thus we obtain the following equation for finding the angle α in terms of the known magnitudes F_1, F_2 and γ:

$$-2F_1.\sin\alpha.\cos\alpha+2F_2.\sin(\gamma-\alpha).\cos(\gamma-\alpha)=0;$$

which may also be put in the following form:

$$\frac{F_1}{\sin 2(\gamma-\alpha)} = \frac{F_2}{\sin 2\alpha}.$$

Moreover, since $P = (P+Q)-Q$, we find:

$$P = F_1(\cos^2\alpha - \sin^2\alpha) + F_2 \left\{\cos^2(\gamma-\alpha) - \sin^2(\gamma-\alpha)\right\}$$
$$= F_1.\cos 2\alpha + F_2.\cos 2(\gamma-\alpha);$$

and if in this formula we substitute the value

$$F_2 = \frac{\sin 2\alpha}{\sin 2(\gamma-\alpha)} F_1,$$

we shall find:

$$P = \frac{\sin 2\gamma}{\sin 2(\gamma-\alpha)} F_1.$$

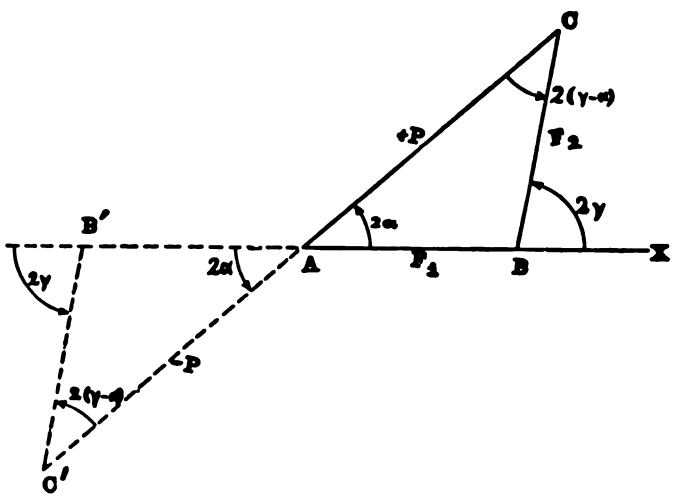


Fig. 157.—Graphical mode of finding cylindrical component (P) of sphero-cylinder equivalent to two obliquely crossed cylinders of powers F_1 and F_2.

Hence,

$$\frac{F_1}{\sin 2(\gamma-\alpha)} = \frac{F_2}{\sin 2\alpha} = \frac{P}{\sin 2\gamma};$$

which at once suggests an elegant and simple graphical solution of the problem. For, evidently, according to the above relations, the magnitudes denoted by F_1, F_2 and P may be represented in a diagram (Fig. 157) by the sides of a triangle whose opposite angles are $2(\gamma-\alpha)$, 2α and $(180°-2\gamma)$, respectively. Hence the rule is as follows:

On any straight line lay off a segment AB to represent, according to a certain scale, the magnitude of the refracting power F_1; and let X designate the position of a point on AB produced beyond B. Construct the $\angle$XBC equal to *twice* the angle between the axes of the two given cylindrical components ($\angle XBC=2\gamma$); and along the side BC of this angle lay off the length BC to represent the magnitude of the refracting power F_2. Then the straight line AC will represent on the same scale the magnitude of the refracting power P of the cylindrical member of the equivalent sphero-cylindrical lens, and the $\angle BAC=2\alpha$ will be equal to *twice* the angle between the cylindrical axes of the surfaces whose powers are denoted by F_1 and P. For calculating the values of P, Q and α, we have by trigonometry the following system of formulæ:

$$P=+\sqrt{F_1^2+F_2^2+2F_1.F_2.\cos 2\gamma},$$

$$Q=\frac{F_1+F_2-P}{2},$$

$$\tan 2\alpha=\frac{F_2.\sin 2\gamma}{F_1+F_2.\cos 2\gamma};$$

which will be found to be applicable in all cases, whether the signs of F_1, F_2 are like or unlike.

There is, to be sure, another solution also, in which the cylindrical axis of the sphero-cylindrical lens is inclined to the cylindrical axis of the cylinder of power F_1 at the angle $(90°+\alpha)$. For if the refracting power Q of the spherical member is assumed to be the maximum (instead of the minimum) refracting power of the sphero-cylindrical combination, then $(P+Q)$ will be the minimum power in a section at right angles to the axis of the cylinder; and in this

case the refracting power of the cylindrical component will be represented by the dotted line AC′ in Fig. 157 which is equal to AC in length but opposite to it in direction. In fact, in this case the formulæ for P and Q will be as follows:

$$P = -\sqrt{F_1^2+F_2^2+2F_1.F_2.\cos 2\gamma},$$

$$Q = \frac{F_1+F_2+P}{2}.$$

This result could have been obtained from the first result by transposing; for, according to § 115, Q sph. $\bigcirc$ P cyl., ax. ϕ is equivalent to $(P+Q)$ sph. $\bigcirc$ $-P$ cyl. ax. $(\phi \pm 90°)$, where the symbols P and Q denote here the powers of the first combination.

Moreover, since Q sph. $\bigcirc$ P cyl., ax. ϕ is equivalent also to $(P+Q)$ cyl., ax. ϕ $\bigcirc$ Q cyl., ax. $(\phi \pm 90°)$, two obliquely crossed cylinders may be replaced by a cross-cylinder of powers $(P+Q)$ and Q. In fact, since

$$(P+Q)+Q = F_1+F_2,$$

$$(P+Q)-Q = \sqrt{F_1^2+F_2^2+2F_1.F_2.\cos 2\gamma},$$

it follows that:

$$(P+Q)Q = F_1.F_2.\sin^2\gamma;$$

so that this formula will give us the product of the powers of the equivalent cross-cylinder, and since their sum $P+2Q = F_1+F_2$, the values of $(P+Q)$ and Q may be obtained independently, without first finding the value of P.

The following numerical example will serve to illustrate the use of the formulæ:

Given a combination of obliquely crossed cylinders as follows:

$$+4 \text{ cyl., ax. } 20° \bigcirc -2.75 \text{ cyl., ax. } 65°;$$

let it be required to find the equivalent sphero-cylinder and also the equivalent cross-cylinder.

We must put $F_1 = +4$, because F_1 denotes the power of the cylinder whose axis-slope is the smaller of the two. Then $F_2 = -2.75$ and $\gamma = (65° - 20°) = 45°$. Substituting these values, we find:

$$P = +4.85, \quad Q = -1.8, \quad \alpha = -17°16'.$$

Accordingly, the given combination is equivalent to one of the three following:

$+4.85$ cyl., ax. $2° 44' \bigcirc -1.8$ sph.;
-4.85 cyl., ax. $92° 44' \bigcirc +3.05$ sph.;
$+3.05$ cyl., ax. $2° 44' \bigcirc -1.8$ cyl., ax. $92° 44'$.

If $\gamma = 90°$, then $P = F_1 - F_2$, $Q = F_2$ and $\alpha = 0°$, or $P = F_2 - F_1$, $Q = F_1$ and $\alpha = 0°$; so that we can write:

F_1 cyl., ax. $\phi \bigcirc F_2$ cyl., ax. $(\phi \pm 90°)$

is equivalent to

F_1 sph. $\bigcirc (F_2 - F_1)$ cyl., ax. $(\phi \pm 90°)$

or

F_2 sph. $\bigcirc (F_1 - F_2)$ cyl., ax. ϕ;

exactly as found in § 115.

PROBLEMS

1. The radius of a convex cylindrical refracting surface separating air from glass ($n = 1.5$) is $8\frac{1}{3}$ cm. What is its refracting power in a normal section inclined to the axis of the cylinder at an angle of 60°? Ans. $+4.5$ dptr.

2. A curved refracting surface separates air and glass ($n' : n = 3 : 2$), and the radii of greatest and least curvature at a point A on the surface are $r_y = +10$ cm. and $r_z = +5$ cm. Find the interval between the two principal image-points corresponding to an object-point lying on the normal to the surface at A in front of the surface and at a distance of 30 cm. from it. Ans. 67.5 cm.

3. The principal refracting powers of a thin astigmatic lens surrounded by air are denoted by F_y and F_z. The principal image-points corresponding to an axial object-point M are designated by M_y' and M_z'. If the optical center of the lens is designated by A, and if we put $U = 1/u$, where $u = AM$, then

$$M_y'M_z' = \frac{F_y - F_z}{(F_y + U)(F_z + U)}.$$

4. The refracting powers of a thin astigmatic lens in the two principal sections are +3 and +5 dptr. The lens is made of glass of index 1.5. Find the radii of the two surfaces for each of the following forms: (*a*) Cross-cylinder; (*b*) Sphero-cylinder; *c*) Plano-toric.

Ans. (*a*) Double convex cross-cylinder, radii 10 and 16 $\frac{2}{3}$ cm.; (*b*) Double convex sphero-cylinder, radius of sphere 16 $\frac{2}{3}$ cm., radius of cylinder 25 cm.; or convex meniscus sphero-cylinder, radius of sphere 10 cm., radius of cylinder 25 cm.; (*c*) Radii of toric surface 10 and 16 $\frac{2}{3}$ cm.

5 The principal refracting powers of a thin lens are +4 and −5 dptr. If the refracting power in an oblique normal section is +2 dptr., what will be its refracting power in a normal section at right angles to the first? and what is the angle of inclination of the +2 section to the +4 section?

Ans. −3 dptr.; 28° 7′ 32″.

6. Two cylinders each of power +1.18 dptr. are combined with their axes inclined to each other at an angle of 32° 3′ 50″. Show that the combination is equivalent to +0.18 sph. ⊂ +2 cyl., axis midway between the axes of the two given cylinders.

7. Show that

+2 cyl., ax. 0° ⊂ −3 cyl., ax. 53° 26′ 14″

is equivalent to

−2.53 sph. ⊂ +4.06 cyl., ax. −22° 30′.

8. Transpose

−1.25 cyl., ax. 20° ⊂ +3.25 cyl., ax. 53° 41′ 24.25″

into the equivalent sphero-cylinder.

Ans. −0.5 sph. ⊂ + 3 cyl., ax. 65°,
or + 2.5 sph. ⊂ −3 cyl., ax. 155°.

9. Transpose

+9.5 cyl., ax. 0° ⊂ +10 cyl., ax. 57° 40′ 45″

into the equivalent sphero-cylinder.

Ans. +4.53 sph. ⊂ +10.43 cyl., ax. 30°,
or +14.96 sph. ⊂ −10.43 cyl., ax. 120°.

10. Find the sphero-cylindrical equivalent of

$+2$ cyl., ax. $20° \supset +3$ cyl., ax. $70°$.

Ans. $+0.85$ sph. $\supset +3.3$ cyl., ax. $51° 42'$,

or $+4.15$ sph. $\supset -3.3$ cyl., ax. $141° 42'$.

11. Transpose

-1.75 cyl., ax. $120° \supset +1.25$ cyl., ax. $135°$

into the equivalent cross-cylinder.

Ans. $+0.207$ cyl., ax. $98° 30' \supset -0.707$ cyl., ax. $8° 30'$.

12. Transpose $+4$ cyl., ax. $80° \supset -2$ cyl., ax. $120°$ into the equivalent cross-cylinder.

Ans. $+3.075$ cyl., ax. $65° 50' \supset -1.075$ cyl., ax. $155° 50'$.

CHAPTER X

GEOMETRICAL THEORY OF THE SYMMETRICAL OPTICAL INSTRUMENT

117. Graphical Method of tracing the Path of a Paraxial Ray through a Centered System of Spherical Refracting Surfaces.—Nearly all optical instruments consist of a combination of transparent, isotropic media, each separated from the next by a spherical (or plane) surface; the centers of these surfaces lying all on one and the same straight line called the *optical axis* of the centered system of spherical surfaces, which is an axis of symmetry. In a symmetrical optical instrument of this kind it is sufficient to investigate the procedure of paraxial rays in any meridian plane containing the axis.

The indices of refraction of the media will be denoted by n_1, n_2, etc., named in the order in which they are traversed by the light; so that if m denotes the number of refracting surfaces, the index of refraction of the last medium into which the rays emerge after refraction at the mth surface will be n_{m+1}. The indices of refraction of the two media which are separated by the kth surface (where k denotes any integer between 1 and m, inclusive) will be n_k and n_{k+1}. The vertex and center of the kth surface will be designated by A_k and C_k, respectively; and the radius of this surface will be denoted by $r_k = A_kC_k$. Moreover, if M_k, M_{k+1} designate the positions of the points where a paraxial ray crosses the axis before and after refraction, respectively, at the kth surface, these points will be a pair of conjugate axial points with respect to this surface; and the points M_1, M_{m+1} will, therefore, be a pair of conjugate axial points with respect

to the entire centered system of m spherical refracting surfaces.

The accompanying diagram (Fig. 158) represents a meridian section of an optical system of this kind. The straight line M_1B_1 represents the path of a paraxial ray in the first medium (n_1) which crossing the axis at M_1 meets the first surface (y_1) in the point marked B_1. Similarly, the path

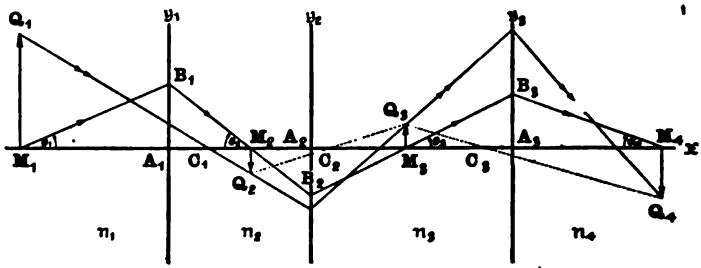


FIG. 158.—Path of paraxial ray through centered system of spherical refracting surfaces.

of the ray from the first surface to the second surface is shown by the straight line B_1B_2 which crosses the axis at M_2. Thus, the entire course of the ray is shown by the broken line $M_1B_1B_2B_3M_4$ which is bent in succession at each of the incidence-points B_1, B_2, B_3 (supposing that $m=3$, as represented in the diagram).

The figure shows also the path of another paraxial ray, emanating from an object-point Q_1 near the optical axis but not on it and represented here as lying perpendicularly above M_1. This ray is the ray which leaves Q_1 along a straight line which passes through the center C_1 of the first refracting surface and also through the point Q_2 which is conjugate to Q_1 with respect to this surface. This point Q_2 can be located by determining the point of intersection of the straight line Q_1C_1 with the straight line M_2Q_2 drawn perpendicular to the axis at M_2. Similarly, the point Q_3 conjugate to Q_2 with respect to the second refracting surface will be at

the point of intersection of the straight line Q_2C_2 with the straight line drawn perpendicular to the axis at M_2; and so on from one surface to the next. Provided, therefore, we know the path of one paraxial ray through the system, it is easy to construct the path of a second ray.

But the best graphical method of tracing the path of a paraxial ray through a centered system of spherical refracting surfaces consists in applying the construction described

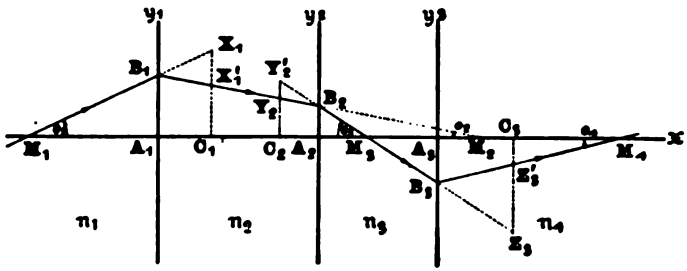


FIG. 159.—Graphical method of tracing path of paraxial ray through centered system of spherical refracting surfaces.

in § 76, as follows: If the straight line M_1B_1 (Fig. 159) representing the path of the ray in the first medium meets the perpendicular erected to the optical axis at the center C_1 in the point X_1, and if on this perpendicular a second point X_1' is taken such that $C_1X_1 : C_1X_1' = n_2 : n_1$, then the straight line B_1X_1' will determine the path B_1B_2 of the ray in the second medium. Draw C_2Y_2 parallel to C_1X_1, and let Y_2 designate the point of intersection of the straight lines B_1B_2 and C_2Y_2; and on C_2Y_2 take a point Y_2' such that $C_2Y_2 : C_2Y_2' = n_3 : n_2$, and draw the straight line $Y_2'B_2$ meeting the third refracting surface in B_3 and intersecting in Z_3 the straight line drawn through C_3 parallel to C_2Y_2. If on C_3Z_3 a point Z_3' is taken such that $C_3Z_3 : C_3Z_3' = n_4 : n_3$, then the straight line B_3Z_3' will determine the path of the ray after refraction at the third surface. This process is to be repeated until the ray has been traced into the last medium.

118. Calculation of the Path of a Paraxial Ray through a Centered System of Spherical Refracting Surfaces.—Obviously, just as in the case of a single spherical refracting surface (§ 80), any figure lying in a plane in the object-space perpendicular to the optical axis of a centered system of spherical refracting surfaces will be reproduced by means of paraxial rays by a similar figure in the image-space also lying in a plane perpendicular to the optical axis.

Moreover, if we put

$$A_kM_k = u_k, \qquad A_kM_{k+1} = u_k',$$

the abscissa-formula (§ 78) for the kth surface may be written:

$$\frac{n_{k+1}}{u_k'} = \frac{n_k}{u_k} + \frac{n_{k+1} - n_k}{r_k}.$$

If also we employ the symbol

$$d_k = A_kA_{k+1}$$

to denote the distance of the vertex of the $(k+1)$th surface from that of the kth surface or the so-called *axial thickness* of the $(k+1)$th medium, then, evidently:

$$u_{k+1} = u_k' - d_k;$$

which enables us to pass from one surface to the next.

If in these so-called recurrent formulæ we give k in succession the values $k = 1, 2, \ldots, (m-1)$, and if also in the first formula we put finally $k = m$, we shall obtain $(2m-1)$ equations; and if the constants of the system are all known, that is, if the values of all the magnitudes denoted by n, r and d are given, together with the initial value u_1, which denotes the abscissa of the axial object-point, these $(2m-1)$ equations will enable us to determine the value of each of the u's in succession. The position of the image point M_{m+1} conjugate to the axial object-point M_1 will have been ascertained when we have found the value of the abscissa u_m'.

The *secondary focal point* of the system is the point F' where a paraxial ray which is parallel to the axis in the first medium crosses the axis in the last medium; and if we put $u_1 = \infty$, then $u_m' = A_mF'$ will be the abscissa of the second-

ary focal point with respect to the vertex of the last surface. Similarly, the *primary focal point* is the point F where a paraxial ray must cross the axis in the first medium if it is to emerge in the last medium in a direction parallel to the axis. In this case, therefore, we must put $u_m' = \infty$ and solve for $u_1 = A_1F$ in order to obtain the abscissa of the focal point F with respect to the vertex of the first surface of the system.

The *focal planes* are the planes at right angles to the axis at the focal points F, F'.

Moreover, if we put $y_k = M_kQ_k$, then according to the formula for the *lateral magnification* in a spherical refracting surface (§ 82), we can write for the kth surface:

$$\frac{y_{k+1}}{y_k} = \frac{n_k}{n_{k+1}} \cdot \frac{u_k'}{u_k};$$

and if we give k all integral values from $k=1$ to $k=m$, we shall obtain m equations, one for each surface, wherein the denominator of the ratio on the left-hand side of each of these proportions will be the same as the numerator of the corresponding ratio in the preceding one of the series. Hence, if we multiply together all of these equations, and if, finally, we put

$$y = y_1, \quad y' = y_{m+1}, \quad n = n_1, \quad n' = n_{m+1},$$

we shall obtain:

$$\frac{y'}{y} = \frac{n}{n'} \frac{u_1 . u_2' \ldots u_m'}{u_1 . u_2 \ldots u_m};$$

which may be written also:

$$\frac{y'}{y} = \frac{n}{n'} \prod_{k=1}^{k=m} \frac{u_k'}{u_k},$$

where the symbol Π placed in front of an expression in this way means merely that the continued product of all terms of that type is to be taken. Thus having found the values of all the u's, both primed and unprimed, we can calculate by this formula the *lateral magnification* produced by the

entire centered system of spherical refracting surfaces for any given position of the object-point.

Moreover, for the kth surface the so-called SMITH-HELMHOLTZ formula (§ 86) will have the form:

$$n_k.y_k.\,\theta_k = n_{k+1}.y_{k+1}.\,\theta_{k+1},$$

where $\theta_k = \angle A_k M_k B_k$; and if here also we give k all values in succession from $k=1$ to $k=m$, we shall obtain:

$$n_1.y_1.\,\theta_1 = n_2.y_2.\,\theta_2 = \ \ldots\ = n_{m+1}.y_{m+1}.\,\theta_{m+1};$$

and finally:

$$n'.y'.\,\theta' = n.y.\,\theta,$$

where n, n' and y, y' have the same meanings as above, and

$$\theta = \theta_1, \qquad \theta' = \theta_{m+1}.$$

119. The so-called Cardinal Points of an Optical System. The methods which have just been explained, although perfectly simple in principle, involve a more or less tedious process of tracing the path of a paraxial ray from one surface to the next throughout the entire system. We have now to explain the celebrated theory of GAUSS (1777–1855) which was developed (1841) in order to avoid as much of this labor as possible, by keeping steadily in view the fundamental relations between the object-space and the image-space. It is easy to show that the imagery produced by a symmetrical optical instrument in the vicinity of the axis is completely determined so soon as we know the positions of the focal points and one pair of conjugate points on the axis, together with the ratio of the indices of refraction of the first and last media of the system. However, for this purpose certain pairs of conjugate axial points are distinguished above others on account of their simple geometrical relations; and of these the most important are the *principal points* and the *nodal points*. These two pairs of conjugate points, together with the focal points, are sometimes called the *cardinal points* of the optical system. We shall explain now how these points are defined.

(1) *The Focal Planes and the Focal Points.*—In every centered system of spherical refracting surfaces there are

two (and only two) transversal planes at right angles to the axis which are characterized by the following properties:

A bundle of paraxial object-rays which all meet in a point in one of these planes (called the primary focal plane) will emerge from the system as a cylindrical bundle of parallel image-rays; and, similarly, *a cylindrical bundle of parallel object-rays will emerge from the system as a bundle of image-rays which all meet in a point in the other one of these planes (called the secondary focal plane).* The points in which these focal planes are pierced by the axis are *the primary and secondary focal points* F and F′, respectively.

(2) *The Principal Planes and the Principal Points.*—Again, in every symmetrical optical system there is one (and only one) pair of conjugate transversal planes characterized by the property, that *in these planes object and image are congruent;* and, therefore, *any straight line drawn parallel to the axis will intersect these planes in a pair of conjugate points.* These are the so-called *principal planes,* one belonging to the object-space (*the primary principal plane*) and the other belonging to the image-space (*the secondary principal plane*). The points H, H′ where the optical axis crosses the principal planes are the *principal points* of the system. Attention was first directed to these points by MOEBIUS in 1829, but it was GAUSS who recognized their significance for the development of simple and convenient general formulæ in the theory of optical imagery.

In the principal planes the lateral magnification is unity, that is, $y'=y$. (And hence the principal planes and principal points are called also, especially by English writers, the *unit planes* and the *unit points.*) Consider, for example, the case of a single spherical refracting surface, for which we found (§ 85)

$$\frac{y'}{y}=\frac{f}{f+u}=\frac{f'+u'}{f'}.$$

If we put $y'=y$, we find $u'=u=0$; which means that *the principal points of a spherical refracting surface coincide with each other at the vertex of the surface* (§ 81). We saw likewise

that these points coincided with each other at the optical center of an infinitely thin lens (§ 94).

A useful rule is as follows:

To any ray in one region (object-space or image-space) which goes through the focal point belonging to that region,

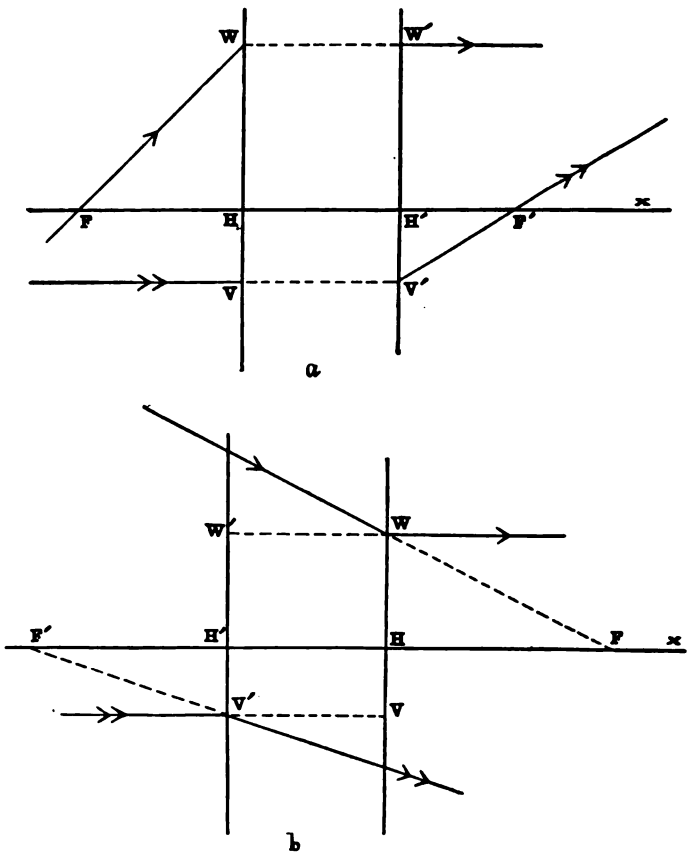


FIG. 160, *a* and *b*.—Focal points (F,F') and principal points (H, H') of (*a*) convergent and (*b*) divergent optical system.

there will correspond a ray in the other region which is parallel to the axis, and the rectilinear portions of the path of

the ray in these two regions will intersect in a point lying in the principal plane of that region to which the focal point in question belongs; as is illustrated in the accompanying diagrams at W and at V′ (Fig. 160, *a* and *b*).

(3) *The Nodal Planes and the Nodal Points.*—Finally, in every centered system of spherical refracting surfaces there is also a pair of conjugate transversal planes characterized by the property, that *the angle between any pair of object-*

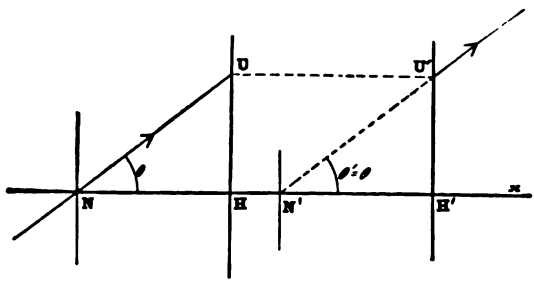


Fig. 161.—Principal points (H, H′) and nodal points (N, N′).

rays which intersect in a point lying in the so-called primary nodal plane will be exactly equal to the angle between the corresponding pair of image-rays which meet in the conjugate point of the secondary nodal plane. The *nodal points* N, N′ where the axis meets these planes were remarked first by Moser in 1844, but they were brought into prominence through the work of Listing (1845) with whose name therefore they are generally associated. The distinguishing feature of this pair of conjugate axial points is that *object-ray and image-ray cross the axis at the nodal points at exactly the same slope.* For example, if the straight line NU (Fig. 161) represents the path of an object-ray which crosses the axis at the primary nodal point and meets the primary principal plane in the point marked U, the path of the corresponding image-ray will be represented by a straight line N′U′ which is drawn parallel to NU and which meets the secondary prin-

cipal plane in the point marked U', so that if $\angle HNU = \theta$, $\angle H'N'U' = \theta'$, then $\theta' = \theta$.

Obviously, the quadrilateral NUU'N' is a parallelogram, and hence H'N' = HN; that is, *the step from one of the principal points to the corresponding nodal point is identical with the step from the other principal point to its corresponding nodal point.* The nodal points, therefore, lie always on the same side of the corresponding principal points and at equal distances from them. If the primary nodal point and principal point coincide, the same will be true of the secondary nodal point and principal point. Moreover, since NN' = UU' = HH', the interval between the nodal planes is precisely the same as the interval between the principal planes.

If in the SMITH-HELMHOLTZ formula (§ 118) we put $\theta' = \theta$, we find for the lateral magnification in the nodal planes of a centered system of spherical refracting surfaces

$$\frac{y'}{y} = \frac{n}{n'},$$

where n and n' denote the indices of refraction of the first and last media, respectively. Applying this result to the case of a single spherical refracting surface, we obtain for the nodal points N, N' the conditions $u' = u = r$, that is, AN' = AN = AC. Consequently, the nodal points of a spherical refracting surface coincide with each other at the center C of the surface; as might have been inferred at once from the fact that a central ray is not deviated by refraction at a spherical surface.

(4) Various writers on optics have distinguished other pairs of conjugate axial points besides the principal points and nodal points, but none of these can be said to have achieved a permanent place in the literature of the subject. We may mention the so-called *negative principal points*, introduced by TOEPLER in 1871, which are characterized by the fact that *for this pair of points the lateral magnification is equal to* -1; that is, $y' = -y$, so that the image is inverted and of same size as object. Professor S. P. THOMPSON, hav-

ing this property in view, has re-named them much more happily the *symmetric points* of the optical system.

120. Construction of the Image-Point Q′ conjugate to an Extra-axial Object-Point Q.—If the principal planes and focal planes have been determined, it will not be necessary to trace the path of a ray in the interior of the system. Sup-

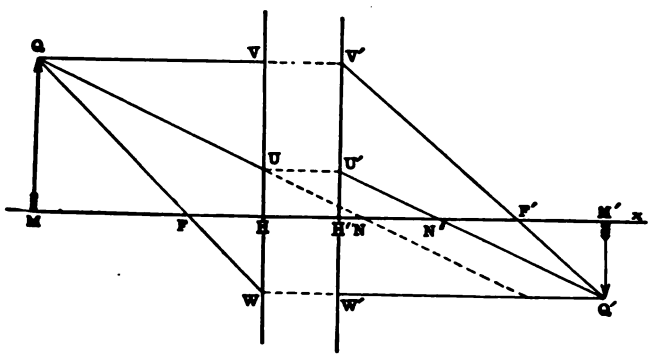


Fig. 162.—Construction of image-point Q′ conjugate to object-point Q in an optical system.

pose, for example, that Q (Fig. 162) designates the position of an object-point not on the axis; the position of the point Q′ conjugate to Q may be constructed as follows:

Through Q draw a straight line QV parallel to the axis meeting the secondary principal plane in the point marked V′ and also another straight line QF meeting the primary principal plane in the point marked W. The required point Q′ will be found at the point of intersection of the straight line V′F′ with the straight line WQ′ drawn parallel to the axis. The feet of the perpendiculars let fall from Q, Q′ on to the axis will locate also a pair of conjugate axial points M, M′. The construction is seen to be entirely similar to that given in §§ 71, 81 and 92. The case represented in the figure is that of a *convergent optical system*, in which parallel object rays are converged to a real focus at a point in the secondary focal plane. The student should draw for him-

self the corresponding diagram for the case of a *divergent optical system.*

121. Construction of the Nodal Points N, N′.—Having determined the position of the point Q′ conjugate to Q, we can easily locate the positions of the nodal points N, N′. For example, on the straight line WQ′ (Fig. 162) take a point Z such that ZQ′ = HH′, and draw the straight line QZ meeting the primary principal plane in the point U. Draw UU′ parallel to the axis meeting the secondary principal plane in the point U′. Evidently, the straight lines QU and Q′U′ will

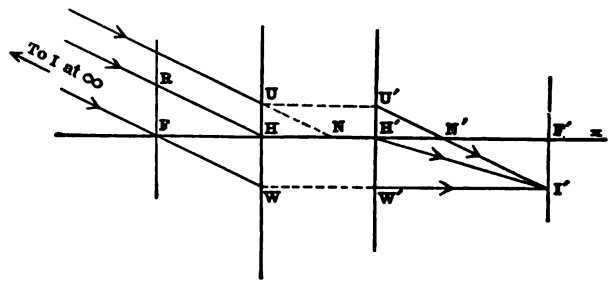


FIG. 163.—Construction of nodal points (N, N′), and proof of relation I′ F′ = FR.

be parallel, and the points where they cross the axis will be the nodal points N, N′ (§ 119).

A simpler way of constructing the nodal points N, N′ is as follows:

Through the primary focal point F draw a straight line FW meeting the primary principal plane in the point marked W, and through W draw a straight line parallel to the axis meeting the secondary focal plane in a point marked I′ in Fig. 163. This point I′ is the image-point of the infinitely distant point I of the straight line FW. The straight line drawn through I′ parallel to FW will meet the axis in the secondary nodal point N′; and the position of the other nodal point N can be found immediately.

The diagram shows also that

$$FH = N'F';$$

whence it follows (§ 119) that

$$F'H' = NF.$$

Accordingly, *the step from one nodal point to the corresponding focal point is identical with the step from the other focal point to its corresponding principal point.* In fact, the three segments of the axis FF′, HN′ and H′N all have a common half-way point.

Incidentally, another useful relation may be seen at a glance in Fig. 163. Let R designate the point where the ray IH which passes through the primary principal point crosses

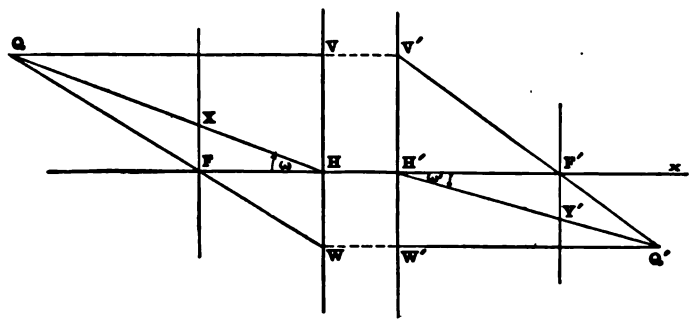


Fig. 164.—Construction of image-point Q′ conjugate to object-point Q in an optical system.

the primary focal plane; the corresponding image-ray will pass through the secondary principal point H′ and cross the secondary focal plane at I′; and, obviously, since FRHW and HWI′F′ are both parallelograms,

$$I'F' = FR;$$

Consequently, *a pair of conjugate rays passing through the principal points* H, H′ *will cross the focal planes at equal distances from the axis, but on opposite sides thereof.*

This result may be utilized in the construction of the point Q′ (Fig. 164) conjugate to the object-point Q. Let X designate the point where the straight line QH crosses the primary focal plane; and take a point Y′ in the secondary focal plane such that F′Y′ = XF. Then the required point Q′

will be at the point of intersection of the straight line H'Y' with either of the straight lines W'Q' or V'F' shown in the figure.

122. The Focal Lengths f, f'.—Let us employ the symbols ω, ω' to denote the slopes of a pair of conjugate rays which pass through the principal points H, H'; thus, in Fig. 164 $\angle FHX = \omega$, $\angle F'H'Y' = \omega'$; and since in the case of paraxial rays we may write ω and ω' in place of $\tan\omega$ and $\tan\omega'$ (see § 63), we have:

$$\frac{FX}{FH} = -\omega, \qquad \frac{F'Y'}{F'H'} = -\omega'.$$

Accordingly, dividing one of these equations by the other, and taking account of the fact that $F'Y' = XF$ (§ 121), we obtain:

$$\frac{FH}{F'H'} = -\frac{\omega'}{\omega}.$$

Since the lateral magnification in the principal planes is equal to $+1$, that is, since $y' = y$ (§ 119), the Smith-Helmholtz formula (§ 118) for the pair of conjugate points H,H' takes the form:

$$n'.\omega' = n.\omega,$$

where n and n' denote the indices of refraction of the first and last media of the optical system.

If, therefore, *the focal lengths of the optical system are defined as the abscissæ of the principal points with respect to their corresponding focal points*, that is, if we put $f = FH$, $f' = F'H'$, where f and f' denote the primary and secondary focal lengths, respectively, then combining the relations found above so as to eliminate the angles ω and ω', we find:

$$\frac{f}{f'} = -\frac{n}{n'};$$

which may be put in words as follows: *The focal lengths of a centered system of spherical refracting surfaces are proportional to the indices of refraction of the first and last media, and are opposite in sign;* except in the single case when the optical system includes *an odd number of reflecting surfaces,*

in which case *the focal lengths will have the same sign* (that is, in this exceptional case, $f/f' = +n/n'$).

It appears, therefore, that the formula,

$$n'.f + n.f' = 0,$$

which was found (§§ 79 and 96) to hold for a single spherical refracting surface and for an infinitely thin lens, expresses, in fact, a perfectly general relation which is true of any centered system of spherical refracting surfaces. Consider, for example, the optical system of the human eye in which the first medium is air ($n = 1$) and the last medium is the

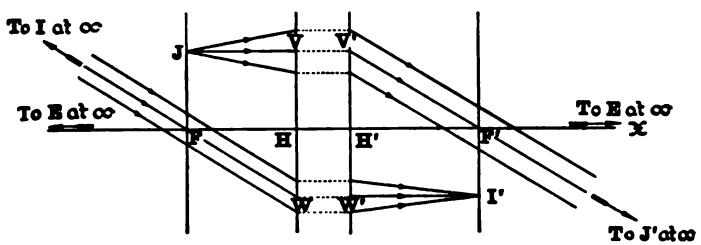


FIG. 165.—Focal lengths (f, f') of an optical system.

so-called vitreous humor whose index of refraction is $n' = 1.336$. In GULLSTRAND's schematic eye (see § 130) the primary focal length is found to be $f = +17.055$ mm., whence, according to the above formula, the secondary focal length is $f' = -22.785$ mm.

In particular, *when the media of object-space and image-space are identical* ($n' = n$), *the focal lengths are equal in magnitude, but opposite in sign* ($f' = -f$). This is the case with most optical systems, since they are usually surrounded by air. According to the definitions of the focal lengths given above, it follows from § 121 that

$$FH = N'F' = f, \quad F'H' = NF = f';$$

and hence we see that *the nodal points* (N, N') *of an optical system surrounded by the same medium on both sides coincide with the principal points* (H, H'); for when $n' = n$, then

$$FH = f = -f' = FN, \quad F'H' = f' = -f = F'N'.$$

The focal lengths of a centered system of spherical refracting surfaces may be defined also exactly as in §§ 83 and 95. If in Fig. 165 we put $\angle HFW = \theta, \angle H'F'V' = \theta'$, we can write:

$$f = FH = \frac{HW}{\tan\theta}, \quad f' = F'H' = \frac{H'V'}{\tan\theta'};$$

and since $HW = F'I'$, $H'V' = FJ$, $\tan\theta = \theta$, $\tan\theta' = \theta'$, we have:

$$f = \frac{F'I'}{\theta}, \quad f' = \frac{FJ}{\theta'}.$$

Accordingly, we may also define the focal lengths as follows: *The focal length of the object-space* (f) *is equal to the ratio of the linear magnitude of an image formed in the focal plane of the image-space to the apparent* (*or angular*) *magnitude of the correspondingly infinitely distant object;* and, similarly, *the focal length of the image-space* (f') *is equal to the ratio of the linear magnitude of an object lying in the focal plane of the object-space to the apparent* (*or angular*) *magnitude of its infinitely distant image.*

The focal lengths may be said, therefore, to measure the *magnifying power* of the optical instrument, for if the apparatus is adapted to an emmetropic eye (§ 153), the image will be formed at infinity, and the magnifying power will be determined by the ratio of the apparent size of the image to the actual size of the object (see Chapter XIII).

123. The Image-Equations in the case of a Symmetrical Optical System.—The image-equations are a system of relations which enable us to find the position of an image-point Q′ (Fig. 162) conjugate to a given object-point Q. The position of the point Q will be given by its two co-ordinates referred to a system of rectangular axes in the object-space in the meridian plane in which the point Q lies. Naturally, the optical axis will be selected as the axis of abscissæ and either the primary focal point F or the primary principal point H as the origin. Thus, if we put

$$FM = x, \qquad HM = u, \qquad MQ = y,$$

the object-point Q will be the point (x, y) or the point (u, y), according as we take the origin at F or H, respectively. Similarly, in the image-space, if we put

$$F'M' = x', \quad H'M' = u', \quad M'Q' = y',$$

the coördinates of Q′ will be denoted by (x', y') or (u', y') according as the origin of this system of axes is at F′ or H′, respectively.

a. The image-equations referred to the focal points F, F′.—The following proportions are obtained from the two pairs of similar triangles FHW, FMQ and F′H′V′, F′M′Q′:

$$\frac{HW}{MQ} = \frac{FH}{FM'} \qquad \frac{M'Q'}{H'V'} = \frac{F'M'}{F'H'};$$

and since

$$HW = M'Q' = y', \quad H'V' = MQ = y, \quad FH = f, \quad F'H' = f',$$

we find immediately:

$$\frac{y'}{y} = \frac{f}{x} = \frac{x'}{f'};$$

whence the coördinates x', y' can be found in terms of the given coördinates x, y and the focal lengths f, f'.

These formulæ, which were obtained formerly for certain simple special cases (§§ 69, 85 and 93) are seen, therefore, to be entirely general and applicable always to any symmetrical optical system. The so-called Newtonian form of the abscissa-relation, viz.,

$$x.x' = f.f',$$

shows that the product of the focal-point abscissæ is constant.

b. The image-equations referred to the principal points H, H′.—Again, the following proportions are derived from the two pairs of similar triangles FHW, QVW and F′H′V′, Q′W′V′:

$$\frac{WV}{HW} = \frac{VQ}{FH} = \frac{HM}{FH}, \qquad \frac{V'W'}{H'V'} = \frac{W'Q'}{F'H'} = \frac{H'M'}{F'H'};$$

and since $WV = WH + HV = Q'M' + MQ = -(y' - y)$ and $V'W' = V'H' + H'W' = QM + M'Q' = (y' - y)$, we find:

$$\frac{y' - y}{y'} = -\frac{u}{f}, \qquad \frac{y' - y}{y} = \frac{u'}{f'}.$$

These relations give the following expressions for the lateral magnification:

$$\frac{y'}{y}=\frac{f}{f+u}=\frac{f'+u'}{f'}=-\frac{f}{f'}\frac{u'}{u}.$$

Clearing fractions, we obtain:

$$f.u'+f'.u+u.u'=0,$$

and dividing through by $u.u'$, we have the well-known abscissa-relation:

$$\frac{f}{u}+\frac{f'}{u'}+1=0;$$

which may also be obtained directly by substituting $x=f+u$, $x'=f'+u'$ in the equation $x.x'=f.f'$.

By means of these formulæ, the coördinates u', y' may be found in terms of the given coördinates u, y and the focal lengths f, f'.

Since $n'.f+n.f'=0$ (§ 122), we have also another expression for the lateral magnification, viz.,

$$\frac{y'}{y}=\frac{n.u'}{n'.u};$$

which has likewise been obtained already in the special case of a single spherical refracting surface (§ 82).

A simple and convenient method of locating the positions of pairs of conjugate axial points is suggested by the abscissa-relation

$$\frac{f}{u}+\frac{f'}{u'}+1=0;$$

which may be put in the following form:

$$\frac{HF}{u}+\frac{H'F'}{u'}=1.$$

Suppose, therefore, that the axial line segment H'F' is shoved along the optical axis until the secondary principal point H' is brought into coincidence with the primary principal point H, and that then the optical axis in the image-space (x') is turned about H until it makes a finite angle with the optical axis in the object-space (x), as represented, for example, in Fig. 166. Through the focal points F and F' draw the

straight lines FS and F′S parallel to H′F′ and HF, respectively, and let S designate their point of intersection. Then any straight line drawn through S will intersect x and x' in a pair of conjugate axial points M, M′; for if we put $u = \mathrm{HM}$ and $u' = \mathrm{H'M'}$ in the equation above, the equation will

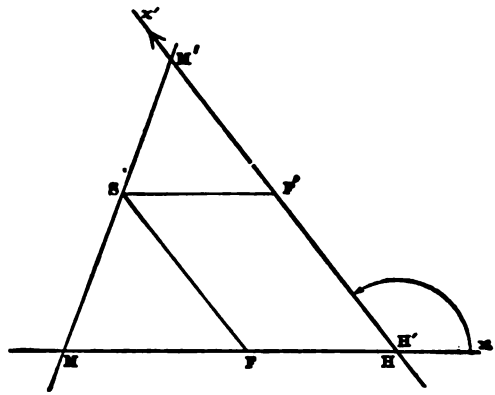


FIG. 166.—Construction of point M′ conjugate to axial object-point M in an optical system.

evidently be satisfied. The vertex S of the parallelogram HF′SF is the center of perspective of the two point-ranges x and x'.

c. The image-equations referred to any pair of conjugate axial points O, O′.

If the origins of the two systems of rectangular axes are a pair of conjugate axial points O, O′ whose distances from the focal points F, F′ are denoted by a, a', respectively, so that $\mathrm{FO} = a$, $\mathrm{F'O'} = a'$; and if we put

$$\mathrm{OM} = z, \qquad \mathrm{O'M'} = z',$$

then

$$x = a + z, \qquad x' = a' + z';$$

and if these values of x and x' are substituted in the equations

$$\frac{y'}{y} = \frac{f}{x} = \frac{x'}{f'},$$

we obtain:

$$\frac{y'}{y}=\frac{f}{a+z}=\frac{a'+z'}{f'}.$$

Since $a.a'=f.f'$, the relation between z and z' may be put in the form:

$$\frac{a}{z}+\frac{a'}{z'}+1=0,$$

where the constants are now a and a' instead of f and f'.

Suppose, for example, that the pair of conjugate axial points O, O′ is identical with the pair of nodal points N, N′; then

$$a=\mathrm{FO}=\mathrm{FN}=-f', \qquad a'=\mathrm{F'O'}=\mathrm{F'N'}=-f;$$

so that *the image-equations referred to the nodal points* will have the following forms:

$$\frac{f'}{z}+\frac{f}{z'}-1=0, \qquad \frac{y'}{y}=\frac{f}{z-f'}=\frac{z'-f}{f'},$$

where $z=\mathrm{NM}$, $z'=\mathrm{N'M'}$.

d. The image-equations in terms of the refracting power and the reduced vergences (see §§ 105 and 106).

The refracting power of the optical system is defined (§ 105) by the relations:

$$F=\frac{n}{f}=-\frac{n'}{f'}$$

where n, n' denote the indices of refraction of the first and last media. Similarly, the reduced vergences (§ 106) with respect to the principal points are:

$$U=\frac{n}{u} \qquad U'=\frac{n'}{u'}.$$

If, therefore, in the image-equations referred to the principal points we eliminate f, f' and u, u' by means of these two pairs of formulæ, we obtain the image-equations in the following exceedingly useful and convenient form:

$$U'=U+F, \qquad \frac{y'}{y'}=\frac{U'}{U'}.$$

If the linear magnitudes are measured in terms of the meter

as unit of length, the magnitudes denoted here by U, U' and F will all be expressed in dioptries (§ 107).

124. The Magnification-Ratios and their Mutual Relations.—(*a*) *The lateral magnification* y. This has already been defined as the ratio of conjugate line-segments lying in planes at right angles to the optical axis. The following expressions were obtained for this ratio in § 123:

$$y = \frac{y'}{y} = \frac{f}{x} = \frac{x'}{f'} = \frac{f}{f+u} = \frac{f'+u'}{f'} = -\frac{f.u'}{f'.u} = \frac{n.u'}{n'.u} = \frac{U}{U'};$$

whence we see that the lateral magnification is a function of the abscissa of the object-point, and that in any optical system it may have any value from $-\infty$ to $+\infty$ depending on the position of the object.

(*b*) *The axial magnification or depth-ratio* x. If x, x' denote the abscissæ with respect to the focal points of a pair of conjugate axial points, and if $x+c$, $x'+c'$ denote the abscissæ of another pair of such points immediately adjacent to the former, then, since

$$x.x' = f.f' = (x+c)\ (x'+c'),$$

and since moreover the product $c.c'$ is a small magnitude of the second order as compared with either of the small factors c or c', and is therefore negligible, we find:

$$c.x' + c.x = 0.$$

The ratio $c' : c$ of small conjugate segments of the axis is called the *axial* or *depth-magnification*. If this ratio is denoted by the symbol x, then, according to the equation above:

$$x = \frac{c'}{c} = -\frac{x'}{x} = -\frac{f.f'}{x^2};$$

so that, whereas *the lateral magnification is inversely proportional to the abscissa* x, *the depth-magnification is inversely proportional to the square of* x. In fact, the relation between the axial magnification and the lateral magnification may be expressed as follows:

$$\frac{x}{y^2} = -\frac{f'}{f} = \frac{n'}{n}.$$

The axial magnification or "depth-elongation" of a small object is proportional to the square of its lateral magnification. If, therefore, we take a series of ordinates, 1, 2, 3, 4, etc. (Fig. 167), all of equal height and at equal intervals apart

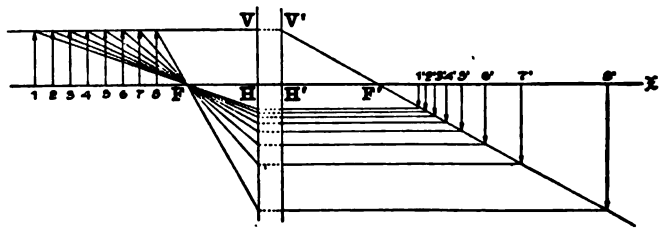


FIG. 167.—Relation between axial or depth-magnification and lateral magnification.

(like a row of telegraph poles), their images will be of unequal heights and at unequal distances apart; but the intervals between the successive images will increase or diminish far more rapidly than the corresponding changes in their heights. Accordingly, the image of a solid object cannot, in general, be similar to the object, but will be distorted, since the dimension parallel to the axis of the optical system is altered very much more than the dimensions at right angles to the axis. This uneven distribution of the images of ob-

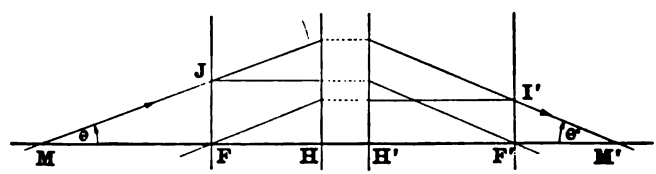


FIG. 168.—Angular magnification or convergence-ratio.

jects at different distances explains "the curious effect noticeable in modern binocular field-glasses of high power, but seen also in opera-glasses and telescopes, in which the successive planes of landscapes seem exaggerated, and flattened almost like the flat scenery of the theater. Thin trees and hedges, for example, seem to occupy definite planes; and

the more distant objects appear to be compressed up toward those in front of them" (Professor S. P. THOMPSON).

(*c*) *The angular magnification or so-called convergence-ratio* z. If the slopes of conjugate rays are denoted by θ, θ', that is, if we put $\theta = \angle \mathrm{FMJ}$, $\theta' = \angle \mathrm{F'M'I'}$ (Fig. 168), where M, M′ designate the points where the ray crosses the axis in the object-space and image-space, respectively, and J and I′ designate the points where it crosses the primary and secondary focal planes, then evidently:

$$\tan\theta = \frac{\mathrm{FJ}}{\mathrm{MF}}, \quad \tan\theta' = \frac{\mathrm{F'I'}}{\mathrm{M'F'}}.$$

But the focal lengths are defined by the equations (§ 122):

$$f = \frac{\mathrm{F'I'}}{\tan\theta'}, \quad f' = \frac{\mathrm{FJ}}{\tan\theta'};$$

and therefore:

$$\tan\theta = \frac{\mathrm{F'I'}}{f}, \quad \tan\theta' = \frac{\mathrm{FJ}}{f'}.$$

Eliminating the intercepts FJ and F′I′, we obtain:

$$z = \frac{\tan\theta'}{\tan\theta} = -\frac{x}{f'} = -\frac{f}{x'},$$

where the ratio $z = \tan\theta' : \tan\theta$ (or $\theta' : \theta$) is called the *angular magnification* or the *convergence-ratio*. It is directly proportional to the abscissa x of the object-point M.

The three magnification-ratios x, y and z are connected by the following relation:

$$\frac{y}{xz} = 1.$$

PROBLEMS

1. Taking the index of refraction of water $= \frac{4}{3}$, show that the sun's rays passing through a globe of water, 6 inches in diameter, will be converged to a focus 6 inches from the center of the sphere.

2. A small object is placed at a distance u from the nearer side of a solid refracting sphere of radius r and of refractive

index n. Show that the distance of the image from the other side of the sphere is

$$u' = \frac{2r(u-r) - n.u.r}{2(n-1)u-(n-2)r},$$

and find the lateral magnification.

3. A luminous point is situated at the first focal point of an infinitely thin symmetric double convex lens made of glass (of index 1.5) and surrounded by air. The radius of each surface is 15 cm. Show that the image formed by rays which have been twice reflected in the interior of the lens before emerging again into the air will be on the other side of the lens at a distance of 2.5 cm. from it.

4. An optical system is composed of two equal double convex lenses. The index of refraction of the glass is $n=1.6202$, and the radii, thicknesses, etc., are as follows:

$$r_1 = -r_4 = 47.92243; \qquad r_3 = -r_2 = 9.39617;$$
$$d_1 = d_3 = 0.2; \qquad d_2 = 2.4287.$$

If an incident paraxial ray crosses the axis at a distance $u_1 = -7.31101$ from the vertex of the first surface, show that the emergent ray will cross the axis at a distance $u'_4 = 33.65725$ from the vertex of the last surface.

5. A. Gleichen in his *Lehrbuch der geometrischen Optik* gives the following data of P. Goerz's "double anastigmat" photographic objective, composed of three cemented lenses, the first being a positive meniscus of crown glass, the second a double concave flint glass lens, and the third a double convex crown glass lens:

Indices of refraction:

$$n_1 = n_5 = 1; \quad n_2 = 1.5117; \quad n_3 = 1.5478; \quad n_4 = 1.6125$$

Radii:

$$r_1 = -0.128965; \quad r_2 = -0.049597; \quad r_3 = +0.196423;$$
$$r_4 = -0.1266629$$

Thicknesses:

$$d_1 = +0.01277; \; d_2 = +0.00664; \; d_3 = +0.02114.$$

Show that the second focal point of this system is at a distance of $+1.111095$ from the vertex of the last surface. (See

scheme for calculation of paraxial ray through a centered system of spherical refracting surfaces, § 181).

6. Define the nodal points N, N′ and show that $FN = -f'$, $F'N' = -f$, where F, F′ designate the positions of the focal points and f, f' denote the focal lengths of the optical system. Under what circumstances are the nodal points identical with the principal points?

7. Derive the image-equations referred to the principal points.

8. Given the positions on the optical axis of the principal points and of the focal points; construct the nodal points. Also, construct the point Q′ conjugate to a given object-point Q. Draw diagrams for convergent and divergent systems.

9. Prove that

$$n'f + nf' = 0,$$

where f and f' denote the focal lengths of the optical system, and n and n' denote the indices of refraction of the first and last media.

10. A small cube is placed on the axis of a symmetrical optical instrument with one pair of its faces perpendicular to the axis. Find the two places where the image of the cube will also be a cube. (Assume that the instrument is surrounded by the same medium on both sides.)

Ans. At the points for which the lateral magnification is $+1$ or -1.

11. An object is placed 3 inches in front of the primary focal plane of a convergent optical system. Show that the image will be one-and-a-half times as large as it was at first if a plate of glass ($n = 1.5$) of thickness 3 inches is interposed in front of the object.

12. Show that the axial magnification at the nodal points has the same value as the lateral magnification in the nodal planes.

13. A symmetrical optical instrument is surrounded by the same medium on both sides. If the images of two small

objects A and B on the axis are formed at A′ and B′, show that the ratio of A′B′ to AB is equal to the product of the lateral magnifications for the pairs of conjugate points A, A′ and B, B′.

14. Show that in a symmetrical optical instrument there are two pairs of conjugate points on the axis for which an infinitely small axial displacement of the object will correspond to an equal displacement of the image; and that the focal points are midway between these points.

15. Show that in a symmetrical optical instrument surrounded by the same medium on both sides there are two points on the axis where object and image will be in the same plane; and that if a denotes the distance between the principal planes, the distance between these two points will be

$$\sqrt{a(a+4f)}.$$

16. In a centered system of m spherical refracting surfaces the vertex of the kth surface is designated by A_k. A paraxial ray crosses the axis before refraction at the first surface at a point M_1 which coincides with the primary focal point F of the optical system. Before and after refraction at the kth surface this ray crosses the axis at M_k and M_{k+1}, respectively. If we put $u_k = A_kM_k$, $u_k' = A_kM_{k+1}$, show that

$$f = \frac{u_2.u_3. \ . \ . \ u_m}{u_1'.u_2'. \ . \ . \ u_{m-1}'} .FA_1,$$

where f denotes the primary focal length of the optical system.

17. If the symbols u_k, u_k', employed in the same sense as in the preceding problem, refer to a paraxial ray which is incident on the first surface of the system in a direction parallel to the optical axis, show that

$$f = \frac{n}{n'} \frac{u_1'.u_2'. \ . \ . \ u_m'}{u_2.u_3. \ . \ . \ u_m}, \quad f' = -\frac{u_1'.u_2'. \ . \ . \ u_m'}{u_2.u_3. \ . \ . \ u_m},$$

where f, f' denote the focal lengths of the system and n, n' denote the indices of refraction of the first and last media.

18. Employing the formulæ of No. 17, determine the focal lengths of a hemispherical lens of glass of refractive index

1.5; and find the positions of the principal planes and the focal planes.

Ans. If r denotes the radius of the curved surface, and if distances are measured from the vertex of this surface, the distances of the focal points are $-2r$ and $+7r/3$, and the distances of the principal points are 0 and $+r/3$. The focal length is twice the length of the radius.

19. If a paraxial ray, proceeding originally in a direction parallel to the axis of a centered system of spherical refracting surfaces (as in No. 17), crosses the axis in the medium of index n_k at a point M_k whose distance from the vertex of the kth surface is $u_k = A_kM_k$ ($U_k = n_k/u_k$), show that

$$F_{1,k} = F_{1,k-1}\,(U_k + F_k)\left(\frac{1}{U_{k-1}+F_{k-1}} - \frac{d_{k-1}}{n_k}\right),$$

where F_k denotes the refracting power of the kth surface, $F_{1,k}$ denotes the refracting power of the system of surfaces bounded by the 1st and kth inclusive ($F_{1,1} = F_1$ and $F_{1,0} = 0$), and $d_{k-1} = A_{k-1}\,A_k$ denotes the axial thickness between the surfaces bounding the medium of index n_k.

CHAPTER XI

COMPOUND SYSTEMS. THICK LENSES AND COMBINATIONS OF LENSES AND MIRRORS

125. Formulæ for Combination of Two Optical Systems in terms of the Focal Lengths.—Suppose that the optical system consists of two parts I and II, each composed of a centered system of spherical refracting surfaces with their optical axes in the same straight line. On a straight line parallel to this common optical axis take two points P, P′ (Fig. 169), which we shall assume to be a pair of conjugate points with respect to the compound system (I+II); and

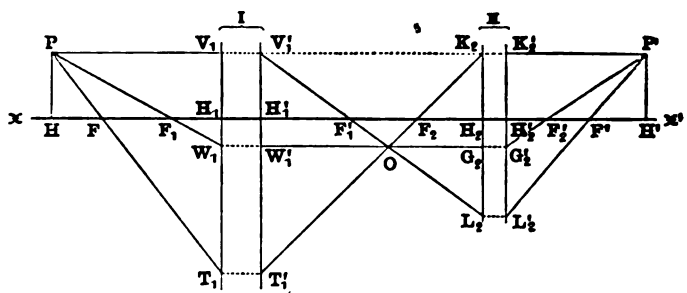


FIG. 169.—Combination of two optical systems. Letters with subscripts refer to component systems; letters without subscripts refer to compound or resultant system.

since these points are on the same side of the optical axis and at equal distances from it, evidently, they must lie in the principal planes of the compound system (§ 119). Accordingly, the feet of the perpendiculars drawn from P, P′ to the optical axis will be the pair of principal points H, H′ of the compound system.

On the optical axis select a point F_1 for the position of

the primary focal point of system I; and select also the positions of the principal points H_1, H_1' and H_2, H_2' of systems I and II, respectively. Through F_1 draw the straight line PW_1 meeting the primary principal plane of system I in the point W_1; take $H_1'W_1' = H_1W_1$, and draw the straight line $W_1'G_2$ parallel to the axis meeting the primary principal plane of system II in the point G_2; take $H_2'G_2' = H_2G_2$, and draw the straight line $G_2'P'$, which must necessarily cross the optical axis at the secondary focal point F_2' of system II.

Let the straight line drawn through P parallel to the optical axis meet the primary and secondary principal planes of system I in the points designated by V_1 and V_1', respectively; and select a point on the optical axis for the position of the secondary focal point F_1' of system I. Through F_1' draw the straight line $V_1'F_1'$ meeting the primary principal plane of system II in L_2; take $H_2'L_2' = H_2L_2$, and draw the straight line $L_2'P'$, which will cross the optical axis in the secondary focal point F' of the compound system.

Let the straight line drawn through P' parallel to the optical axis meet the primary and secondary principal planes of system II in the points K_2 and K_2', respectively; and let O designate the point of intersection of the pair of straight lines $W_1'G_2$ and $V_1'L_2$. The point where the straight line K_2O crosses the optical axis will be the position of the primary focal point F_2 of system II. Let the straight line K_2F_2 meet the secondary principal plane of system I in the point T_1', and take $H_1T_1 = H_1'T_1'$; then the straight line PT_1 will cross the optical axis at the primary focal point F of the compound system.

The diagram constructed according to the above directions represents a perfectly general case. The focal lengths of the component systems are: $f_1 = F_1H_1$, $f_1' = F_1'H_1'$ and $f_2 = F_2H_2$, $f_2' = F_2'H_2'$; and the focal lengths of the compound system are: $f = FH$, $f' = F'H'$. The step from the secondary focal point of the first system to the primary focal point of the system will be denoted by the symbol Δ; thus, $\Delta = F_1'F_2$.

Now if we know the positions on the optical axis of the focal points F_1, F_1' and F_2, F_2' of the two component systems, together with the values of the focal lengths f_1, f_1' and f_2, f_2', it is easy to calculate the positions of the focal points F, F' and the values of the focal lengths f, f' of the compound system; as will now be shown.

The position of the primary focal point F of the compound system may be found from the fact that F and F_2 are a pair of conjugate axial points with respect to system I, and hence (§ 123, *a*):

$$F_1F.\ F_1'F_2 = f_1 f_1'.$$

And, similarly, the position of the secondary focal point F' may be found from the fact that F_1' and F' are a pair of conjugate points with respect to system II, so that

$$F_2'F'.F_2F_1' = f_2 f_2'.$$

Accordingly, the positions of the focal points F, F' with respect to the known points F_1, F_2', respectively, are given by the following formulæ:

$$F_1F = \frac{f_1 f_1'}{\Delta}, \qquad F_2'F' = -\frac{f_2 f_2'}{\Delta}.$$

In order to find the focal lengths f, f', we may proceed as follows:

In the similar triangles FHP, FH_1T_1 we have:

$$\frac{FH}{FH_1} = \frac{HP}{H_1T_1};$$

and since

$$HP = H_2K_2, \qquad H_1T_1 = H_1'T_1',$$

the proportion above may be written:

$$\frac{FH}{FH_1} = \frac{H_2K_2}{H_1'T_1'}.$$

Now from the similar triangles $F_2H_2K_2$, $F_2H_1'T_1'$ we have also:

$$\frac{H_2K_2}{H_1'T_1'} = \frac{F_2H_2}{F_2H_1'};$$

and hence:

$$FH = \frac{F_2H_2}{F_2H_1'} \cdot FH_1.$$

Now $FH_1 = FF_1 + F_1H_1 = -\frac{f_1 \cdot f_1'}{\Delta} + f_1 = -\frac{f_1}{\Delta}(f_1' - \Delta)$;

and $F_2H_1' = F_2F_1' + F_1'H_1' = f_1' - \Delta$.

Accordingly, putting $FH = f$, $F_2H_2 = f_2$, we obtain:

$$f = -\frac{f_1 \cdot f_2}{\Delta},$$

whereby the primary focal length of the compound system may be calculated.

Similarly, from the figure we obtain the relations:

$$\frac{F'H'}{F'H_2'} = \frac{H'P'}{H_2'L_2'} = \frac{H_1'V_1'}{H_2L_2} = \frac{F_1'H_1'}{F_1'H_2};$$

and since $F'H' = f'$, $F_1'H_1' = f_1'$,

$$F'H_2' = F'F_2' + F_2'H_2' = \frac{f_2 \cdot f_2'}{\Delta} + f_2' = \frac{f_2'}{\Delta}(f_2 + \Delta),$$

$$F_1'H_2 = F_1'F_2 + F_2H_2 = f_2 + \Delta,$$

we obtain an analogous expression for the secondary focal length of the compound system, as follows:

$$f' = \frac{f_1' \cdot f_2'}{\Delta}.$$

By varying the interval Δ, which is the common denominator of all these expressions, it is obvious that it is possible with two given component systems to obtain combinations of widely different optical effects. In particular, when F_1' coincides with F_2, so that the interval Δ vanishes, the focal points F, F' will be situated both at infinity, so that the focal lengths f, f' will be infinite also. This is the case, for example, with the optical instrument known as the telescope; and, accordingly, any optical system which transforms a cylindrical bundle of parallel rays into another cylindrical bundle of parallel rays is called a *telescopic* (or *afocal*) *system*. The simplest illustration of such a system is afforded by a single plane refracting surface or by a plane mirror.

126. Formulæ for Combination of Two Optical Systems in terms of the Refracting Powers.—Although the formulæ derived in the preceding section are very simple and convenient, GULLSTRAND's system of formulæ in terms of the refracting powers possesses certain advantages and is even more useful. The latter formulæ may be derived immediately from the former, as will now be shown.

In GULLSTRAND's system the interval between the two component optical systems is expressed, not by Δ, but by the *reduced distance* (§ 104) c of the primary principal point H_2 of system II from the secondary principal point H_1' of system I. Thus, if n_1, n_2 and n_2, n_3 denote the indices of refraction of the first and last media of systems I and II, respectively, then

$$c=\frac{H_1'H_2}{n_2}.$$

The connection between the two magnitudes c and Δ is easily obtained; for since

$$F_1'F_2=F_1'H_1'+H_1'H_2+H_2F_2,$$

we find immediately:

$$\Delta=f_1'+n_2.c-f_2.$$

Now let us introduce the following symbols:

$$F_1=\frac{n_1}{f_1}=-\frac{n_2}{f_1'},\quad F_2=\frac{n_2}{f_2}=-\frac{n_3}{f_2'},\quad F=\frac{n_1}{f}=-\frac{n_3}{f'},$$

where F_1, F_2 denote, therefore, the refracting powers of the component systems and F denotes the refracting power of the compound system (§§ 105 and 123, *d*). Hence, since

$$f_1'=-\frac{n_2}{F_1},\quad f_2=\frac{n_2}{F_2},$$

we may write:

$$\Delta=-\frac{n_2}{F_1.F_2}(F_1+F_2-c.F_1.F_2).$$

Now if this value of Δ is substituted in either of the formulæ

$$f=-\frac{f_1.f_2}{\Delta},\quad f'=\frac{f_1'.f_2'}{\Delta},$$

and if the focal lengths are expressed in terms of the refracting powers, we find:

$$F = F_1 + F_2 - c.F_1.F_2;$$

which is GULLSTRAND's formula for the refracting power of the compound system in terms of the refracting powers of the two component systems and of the interval c between them.

Likewise, if in the formulæ

$$F_1F = \frac{f_1.f_1'}{\Delta}, \quad F_2'F' = -\frac{f_2.f_2'}{\Delta}$$

we eliminate f_1, f_1' and f_2, f_2' and put $\Delta = -n_2\frac{F}{F_1.F_2}$, we obtain for the reduced steps F_1F and $F_2'F'$ the following expressions:

$$\frac{F_1F}{n_1} = \frac{F_2}{F.F_1}, \qquad \frac{F_2'F'}{n_3} = -\frac{F_1}{F.F_2}.$$

The positions of the focal points F, F' of the compound system with respect to H_1, H_2', respectively are obtained as follows:

$$H_1F = H_1F_1 + F_1F = F_1F - n_1/F_1,$$
$$H_2'F' = H_2'F_2' + F_2'F' = F_2'F' + n_3/F_2;$$

and if herein the values of F_1F and $F_2'F'$ are substituted, and if also we note that

$$F - F_1 = F_2(1 - c.F_1), \qquad F - F_2 = F_1(1 - c.F_2),$$

we obtain finally:

$$\frac{H_1F}{n_1} = -\frac{1 - c.F_2}{F}, \qquad \frac{H_2'F'}{n_3} = \frac{1 - c.F_1}{F}.$$

Moreover, since

$$H_1H = H_1F + FH = H_1F + n_1/F,$$
$$H_2'H' = H_2'F' + F'H' = H_2'F' + n_3/F,$$

the GULLSTRAND system of formulæ for the combination of two optical systems may be written as follows:

$$\frac{H_1H}{n_1} = \frac{F_2}{F}.c, \qquad \frac{H_2'H'}{n_3} = -\frac{F_1}{F}.c,$$
$$F = F_1 + F_2 - c.F_1.F_2.$$

Accordingly, if the positions of the principal points H_1, H_1'

and H_2, H_2' of the two component systems, the refracting powers F_1, F_2 and the indices of refraction n_1, n_2 and n_3 are known, we can calculate the reduced interval c and find the refracting power F of the compound system and the positions of the principal points H, H'. We shall see numerous applications of these formulæ in the succeeding sections of this chapter.

127. Thick Lenses Bounded by Spherical Surfaces.— When a centered system of spherical refracting surfaces consists of two surfaces, it constitutes a spherical lens involving three media, viz., the medium of the incident rays (n_1), the medium comprised between the two spherical surfaces, sometimes called the lens-medium (n_2), and the medium of the emergent rays (n_3), which is generally but not necessarily the same as that of the incident rays. Usually, a lens is described by assigning the values of the three indices of refraction and the positions of the centers C_1, C_2 and the vertices A_1, A_2 on the optical axis; the usual data being the radii $r_1 = A_1C_1$, $r_2 = A_2C_2$ and the thickness $d = A_1A_2$. The lens may be regarded, therefore, as a combination of two spherical refracting surfaces whose refracting powers F_1, F_2 are given by the formulæ (§ 105)

$$F_1 = \frac{n_2 - n_1}{r_1}, \quad F_2 = \frac{n_3 - n_2}{r_2}.$$

Since the principal points of a spherical refracting surface coincide with each other at the vertex of the surface (§§ 81 and 119), the interval $c = \frac{H_1'H_2}{n_2} = \frac{A_1A_2}{n_2}$, and therefore

$$c = \frac{d}{n_2}.$$

Accordingly, if, by way of abbreviation, we introduce the special symbol

$$N = n_2\{(n_2 - n_1)r_2 - (n_2 - n_3)r_1\} + (n_2 - n_3)(n_2 - n_1)d$$

to denote a constant of the lens, we obtain, by substituting

the values of F_1, F_2 and c in the formula $F=F_1+F_2-c.F_1.F_2$, the following expression for *the refracting power F of a lens:*

$$F=\frac{N}{n_2.r_1.r_2},$$

where the value of F will be given in dioptries in case the distances r_1, r_2 and d are all measured in meters (§ 107).

The *positions of the principal points* (H, H′) *of a lens* are determined in the same way by the formulæ:

$$\frac{A_1H}{n_1}=-\frac{n_2-n_3}{N}r_1.d, \qquad \frac{A_2H'}{n_3}=-\frac{n_2-n_1}{N}r_2.d;$$

and the *positions of the focal points* (F, F′) may likewise be calculated from the following expressions:

$$\frac{A_1F}{n_1}=-\frac{r_1}{N}\Big\{n_2.r_2+(n_2-n_3)d\Big\}, \quad \frac{A_2F'}{n_3}=\frac{r_2}{N}\Big\{n_2.r_1-(n_2-n_1)d\Big\}.$$

When, as is usually the case, *the lens is surrounded by the same medium on both sides*, we may put

$$n_1=n_3=n, \qquad n_2=n';$$

and then the above formulæ become:

$$N=(n'-n)\{n'(r_2-r_1)+(n'-n)d\};$$

$$F=\frac{N}{n'.r_1.r_2};$$

$$\frac{A_1H}{n}=-\frac{n'-n}{N}r_1.d, \qquad \frac{A_2H'}{n}=-\frac{n'-n}{N}r_2.d;$$

$$\frac{A_1F}{n}=-\frac{r_1}{N}\Big\{n'.r_2+(n'-n)d\Big\}, \quad \frac{A_2F'}{n}=\frac{r_2}{N}\Big\{n'.r_1-(n'-n)d\Big\}.$$

The *nodal points* (N, N′) *of a lens* surrounded by the same medium on both sides coincide with the principal points (§ 122).

The positions of the focal points and principal points may be exhibited in the case of a thick convergent lens in the following manner, as described in GRUMBRECHT'S *Handbuch der Physik:*

Two thin plano-lenses, each 4 cm. in diameter, are cemented with Canada balsam to the opposite faces of a glass

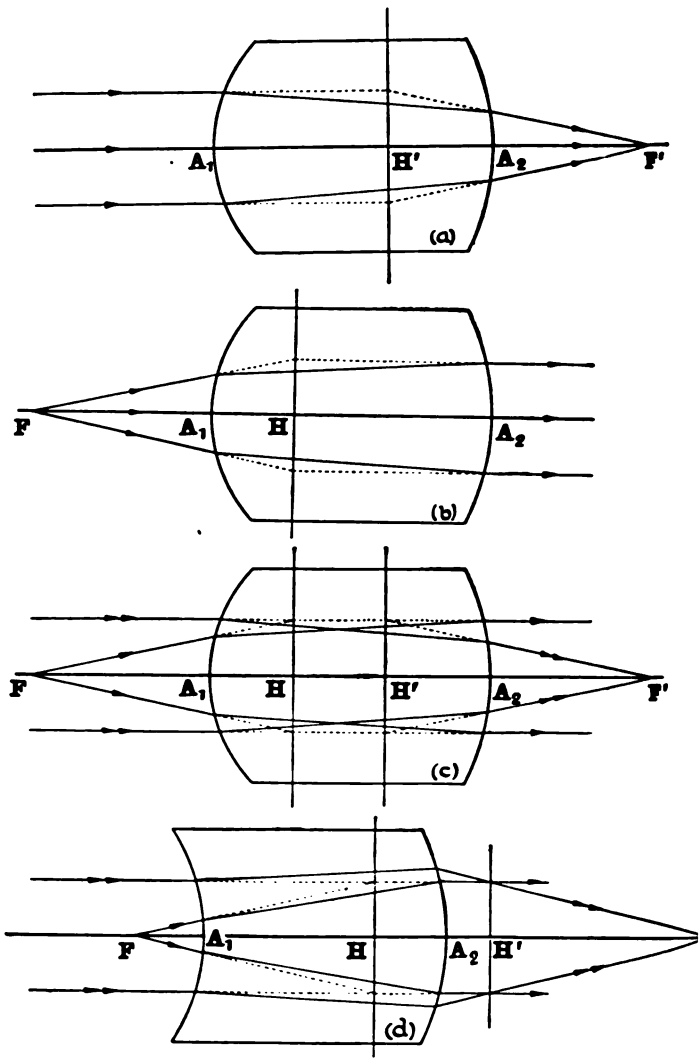


FIG. 170, *a*, *b*, *c*, and *d*.—Double convex lens: (*a*) Location of second point (F′) and principal point (H′); (*b*) Location of first focal point and principal point (H); (*c*) Location of focal points (F, F′) and prin points (H, H′). (*d*) Meniscus convex lens: location of principal p and focal points, showing their unsymmetrical positions with respe the surfaces of the lens.

cube of edge 4 cm. and made of the same glass, so as to form a thick symmetric double convex lens, as represented in Fig. 170, *a*, *b* and *c*. A diaphragm with three parallel horizontal slits is placed in the path of a cylindrical beam of parallel rays so as to separate it into three smaller beams, and the lens is adjusted so that the middle beam proceeds along the axis of the lens. The paths of the rays in air can be rendered visible by tobacco-smoke and may be photographed. In this way figures will be obtained similar to those shown in the diagrams. The position of the secondary focal point F′ is shown by the point of convergence of the rays on emergence (Fig. 170, *a*). A point in the second principal plane of the lens may be located by finding the point of intersection of an incident ray parallel to the axis with the corresponding emergent ray (§ 119), as indicated by the dotted lines in the figure; and the second principal point H′ will be at the foot of the perpendicular dropped from this point on to the axis. If the rays are sent through the lens from the opposite side (that is, from right to left in the drawing, Fig. 170, *b*), they will intersect on emergence in the primary focal point F; and the position of the primary principal point H may be found in exactly the same way as above. The two diagrams Figs. 170, *a* and *b*, are combined in one in Fig. 170, *c*. In Fig. 170, *d*, the lens is concave towards the incident light and convex when viewed from the other side; and this figure shows very clearly how the focal points F, F′ and the principal points H, H′ may be both unsymmetrically placed with respect to the lens, although here also we have, as before, FH = H′F′.

128. So-called "Vertex Refraction" of a Thick Lens.— The step from the second vertex (A_2) of a lens to the second focal point (F′), which may be denoted by v, is sometimes called the "back focus" of the lens; that is, $v = A_2F'$. If the lens is surrounded by the same medium (n) on both sides, then $v/n = (1 - c.F_1)/F$, where F denotes the refracting power of the lens, F_1 denotes the refracting power of

the first surface, and $c=d/n'$ denotes the reduced thickness. The reciprocal of this magnitude v/n is called the *vertex refraction* of the lens $\left(\frac{n}{v}=V\right)$ and its relation to the refracting power is given by the formula:

$$V=\frac{F}{1-c.F_1}=\frac{F}{1-\frac{n'-n}{r_1}\cdot\frac{d}{n'}}.$$

If F is given in dioptries, the values of d and r_1 must be expressed in meters; and then the expression above will give the value of V in dioptries. The importance of this function V in the theory of modern spectacle lenses has been pointed out by Von Rohr; it is measured from the second face of the lens because that is the side next the eye. When a lens (with spherical surfaces) is reversed by turning it through 180° around any line perpendicular to its axis, the refracting power F remains the same, whereas the vertex refraction V will be different unless the lens is a symmetric lens or infinitely thin, in which latter case $d=0$ and $V=F$. Thus, whereas the refracting power of a lens is the same whether the light traverses it from one side or the other, the vertex refraction depends essentially on which side of the lens is presented to the incident rays.

129. Combination of Two Lenses.—Let us take the simplest case, and suppose that the system is composed of *two infinitely thin co-axial lenses*, each surrounded by air. Let A_1 and A_2 designate the points where the optical axis meets the two lenses, and let the interval between them be denoted by c; that is, put $c=A_1A_2$. Since the principal points of an infinitely thin lens coincide with each other at the point A where the axis crosses the lens, and since the intervening medium is assumed to be air of index unity, this distance c has here the same meaning as the reduced interval $c=H_1'H_2/n_2$ in the general formulæ of § 126. Accordingly, we may write immediately the following system of formulæ for a combination of two thin lenses of refracting

powers F_1, F_2, surrounded on both sides by air and separated by the distance c:

$$F = F_1 + F_2 - c.F_1.F_2;$$

$$A_1H = \frac{c.F_2}{F}, \quad A_2H' = -\frac{c.F_1}{F};$$

$$A_1F = -\frac{1-c.F_2}{F}, \quad A_2F' = \frac{1-c.F_1}{F}.$$

These formulæ may also be expressed in terms of the focal lengths f_1 and f_2, as follows:

$$f = \frac{f_1.f_2}{f_1+f_2-c},$$

$$A_1H = \frac{f}{f_2}.c, \quad A_2H' = -\frac{f}{f_1}.c, \quad A_1F = -\frac{f(f_2-c)}{f_2}, \quad A_2F' = \frac{f(f_1-c)}{f_1}.$$

The positions of the focal points F, F′ and the principal points H, H′ of a combination of two infinitely thin lenses surrounded by air may be constructed geometrically as follows:

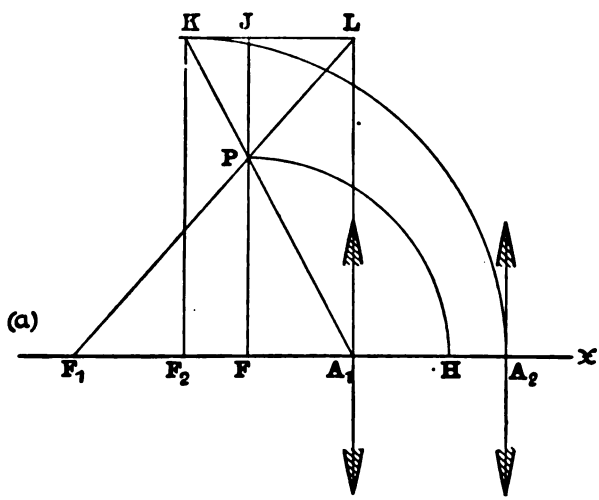


Fig. 171, *a*.—Combination of two thin lenses. Graphical method of determining the positions of the first focal point (F) and principal point (H): Case when both lenses are convex.

Draw a straight line to represent the common axis of the pair of thin lenses, and mark the points A_1 and A_2 (Fig. 171, *a*, *b*, and *c*) where the axis crosses the lenses, and also the positions of the primary focal points F_1 and F_2. Through F_2 draw a straight line perpendicular to the axis, and take on it a point K such that $F_2K = F_2A_2 = f_2$; this point K lying

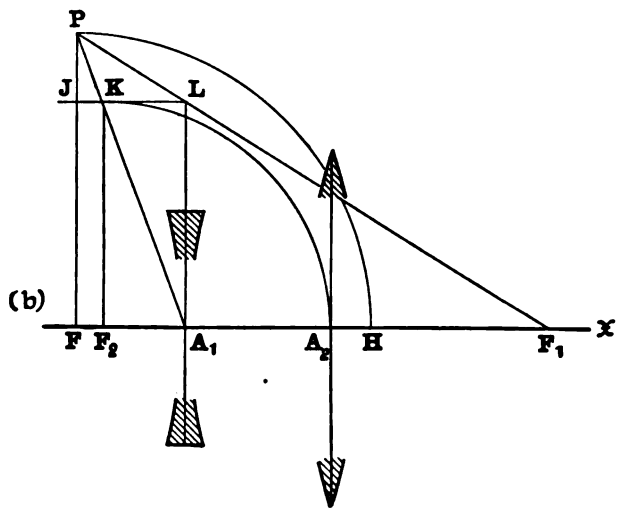


FIG. 171, *b*.—Combination of two thin lenses. Graphical method of determining the positions of the first focal point (F) and principal point (H): Case when first lens is concave and second lens convex.

above or below the axis according as the second lens is convex or concave, respectively. Through K draw a straight line parallel to the axis and through A_1 a straight line perpendicular to the axis; and let L designate the point where these two lines intersect. Moreover, let P designate the point of intersection of the pair of straight lines LF_1 and KA_1. The foot of the perpendicular let fall from P on to the axis will be the primary focal point F of the compound system; and the ordinate FP will be equal to the primary focal length *f* of the compound system; and hence if the quadrant of a

circle is described around F as center with radius FP, it will cut the axis at the primary principal point H, which lies to the right or left of F according as the point P falls above or below the axis.

According to this construction, the points P and K are a pair of conjugate extra-axial points with respect to the

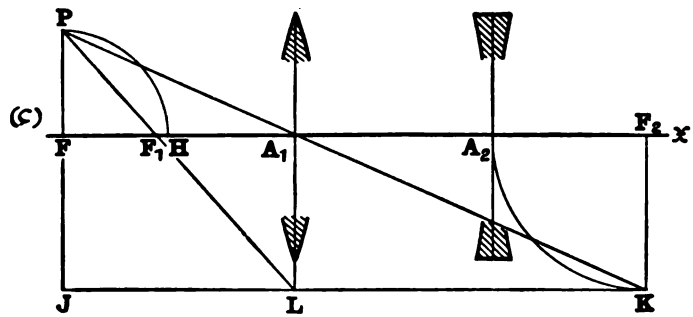


FIG. 171, *c*.—Combination of two thin lenses. Graphical method of determining the positions of the first focal point (F) and principal point (H): Case when first lens is convex and second lens concave.

first lens; so that the construction really consists in locating the object-point P which is imaged by the first lens in the point K. This will help the student to remember the construction.

In order to show that the construction is correct, let J designate the point of intersection of the pair of straight lines FP and LK. Then since JP and FP are corresponding altitudes of the similar triangles PLK and PF_1A_1, we have:

$$\frac{JP}{FP}=\frac{LK}{F_1A_1}=\frac{A_1F_2}{F_1A_1}=\frac{A_1A_2+A_2F_2}{F_1A_1}=\frac{c-f_2}{f_1}.$$

Now $JP=JF+FP=KF_2+FP=FP-f_2$, and therefore:

$$\frac{FP-f_2}{FP}=\frac{c-f_2}{f_1},$$

and if this equation is solved for FP, we find:

$$FP=\frac{f_1 \cdot f_2}{f_1+f_2-c}=f,$$

in agreement with the formula found above. Moreover, in the similar triangles A_1FP and A_1F_2K,

$$\frac{A_1F}{FP}=\frac{A_1F_2}{F_2K};$$

and since $A_1F_2=c-f_2$, $F_2K=f_2$, $FP=f$, we find:

$$A_1F=-\frac{f(f_2-c)}{f_2},$$

which is likewise in agreement with the formula found above.

Similarly, mark the positions of the secondary focal points F_1' and F_2', and through F_1' draw a straight line perpendicular to the optical axis, and take on it a point O such that $F_1'O=F_1'A_1=f_1'$. Through O draw the straight line OR parallel to the axis, and through A_2 a straight line perpendicular to the axis; and let R designate the point where these two lines intersect. Then if Q designates the point of intersection of the straight lines $F_2'R$ and A_2O, that is, if Q is the image of O in the second lens, the secondary focal point F' of the combination will be at the foot of the perpendicular drawn from Q to the optical axis, and the secondary principal point H' will lie on the axis at a distance $F'H'=F'Q$. This construction may be proved in a manner entirely analogous to the proof given above.

130. Optical Constants of Gullstrand's Schematic Eye.—As a further illustration of the use of the formulæ for the

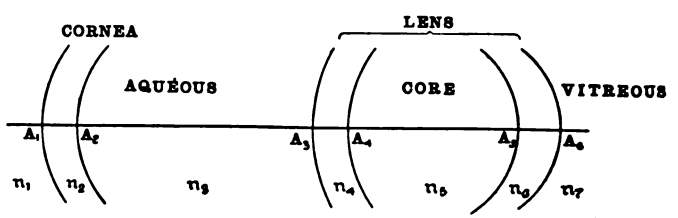


FIG. 172.—Schematic eye.

combination of two optical systems, let us apply them to the calculations of the refracting power (F) of the human eye, together with the positions of the principal points (H,H')

and the focal points (F, F'). For this purpose we shall use the data of GULLSTRAND's schematic eye (in its passive state, accommodation entirely relaxed) which are given in the third edition of HELMHOLTZ's *Handbuch der physiologischen Optik*, Bd. I (Hamburg u. Leipzig, 1909), pages 300 and 301, as follows (see Fig. 172):

Indices of refraction:

Cornea ..... $n_2 = 1.376$
Aqueous and vitreous humors $n_3 = n_7 = 1.336$
Lens ..... $n_4 = n_6 = 1.386$
Lens-core ..... $n_5 = 1.406$

Position of surfaces:

Posterior surface of cornea:	$A_1A_2 = 0.5$	mm.
Anterior surface of lens:	$A_1A_3 = 3.6$	"
" " " lens-core:	$A_1A_4 = 4.146$	"
Posterior " " lens-core:	$A_1A_5 = 6.565$	"
" " " lens:	$A_1A_6 = 7.2$	"

Radii of surfaces:

Anterior surface of cornea:	$r_1 = +\ 7.7$	mm.
Posterior " " "	$r_2 = +\ 6.8$	"
Anterior " " lens:	$r_3 = +10.0$	"
" " " lens-core:	$r_4 = +\ 7.911$	"
Posterior " " " "	$r_5 = -\ 5.76$	"
" " " lens:	$r_6 = -\ 6.0$	"

Consider, first, the cornea-system composed of the anterior and posterior surfaces of the cornea. The refracting power of the anterior surface is:

$$F_1 = \frac{n_2 - n_1}{r_1} = +48.831 \text{ dptr.};$$

and that of the posterior surface is:

$$F_2 = \frac{n_3 - n_2}{r_2} = -5.882 \text{ dptr.}$$

The reduced interval between the two surfaces is:

$$c_1 = \frac{A_1A_2}{n_2} = \frac{0.0005}{1.376}.$$

Hence, if F_{12} denotes the refracting power of the cornea-system, where

$$F_{12}=F_1+F_2-c_1.F_1.F_2,$$

we find:

$$F_{12}=+43.053 \text{ dptr.}$$

The positions of the principal points of the cornea-system are given by the formulæ:

$$\frac{A_1H_{12}}{n_1}=\frac{c_1.F_2}{F_{12}}, \qquad \frac{A_2H_{12}'}{n_3}=-\frac{c_1.F_1}{F_{12}};$$

whence we find:

$$A_1H_{12}=-0.0496 \text{ mm.}, \qquad A_1H_{12}'=-0.0506 \text{ mm.}$$

The lens-system is composed of four refracting surfaces. The first two surfaces form the so-called anterior cortex and the last two surfaces the posterior cortex. The refracting power of the anterior surface of the lens is:

$$F_3=\frac{n_4-n_3}{r_3}=+5 \text{ dptr.};$$

and that of the anterior surface of the lens-core is:

$$F_4=\frac{n_5-n_4}{r_4}=+2.528 \text{ dptr.}$$

The reduced interval between these two surfaces is

$$c_3=\frac{A_3A_4}{n_4}=\frac{0.000546}{1.386}.$$

Hence, if F_{34} denotes the refracting power of the combination, that is, if

$$F_{34}=F_3+F_4-c_3.F_3.F_4,$$

we find: $F_{34}=+7.523$ dptr.

If the principal points of the anterior cortex are designated by H_{34}, H_{34}', then

$$\frac{A_3H_{34}}{n_3}=\frac{c_3.F_4}{F_{34}}, \qquad \frac{A_4H_{34}'}{n_5}=-\frac{c_3.F_3}{F_{34}};$$

whence we obtain:

$$A_1H_{34}=+3.777 \text{ mm.}, \qquad A_1H_{34}'=+3.778 \text{ mm.}$$

so that the principal points of the anterior cortex are coincident with each other.

Proceeding in the same way with the posterior cortex, we have:

$$F_5=\frac{n_6-n_5}{r_5}=+3.472 \text{ dptr.}, \quad F_6=\frac{n_7-n_6}{r_6}=+8.333 \text{ dptr.},$$

$$c_5=\frac{A_5A_6}{n_6}=\frac{0.000635}{1.386};$$

and hence if

$$F_{56}=F_5+F_6-c_5.F_5.F_6,$$

we find: $F_{56}=+11.792$ dptr.

Moreover, since

$$\frac{A_5H_{56}}{n_5}=\frac{c_5.F_6}{F_{56}}, \qquad \frac{A_6H_{56}'}{n_7}=-\frac{c_5.F_5}{F_{56}},$$

we have finally for the positions of the principal points of the posterior cortex:

$$A_1H_{56}=+7.0202 \text{ mm.}, \quad A_1H_{56}'=+7.0198 \text{ mm.};$$

so that H_{56} and H_{56}' may also be regarded as coincident.

If the refracting power of the lens-system as a whole is denoted by L, then

$$L=F_{34}+F_{56}-s.F_{34}.F_{56},$$

where

$$s=\frac{H_{34}H_{56}'}{n_5}=\frac{0.0032422}{1.406};$$

and if P, P′ designate the principal points of the lens-system, then

$$\frac{H_{34}P}{n_3}=\frac{s.F_{56}}{L}, \qquad \frac{H_{56}'P'}{n_7}=-\frac{s.F_{34}}{L}.$$

Accordingly, we find:

$$L=+19.110 \text{ dptr.};$$

$$A_1P=+5.6780 \text{ mm.}, \quad A_1P'=+5.8070 \text{ mm.}$$

Lastly, combining the cornea-system and the lens-system, we obtain for the refracting power of the entire optical system of the eye:

$$F=F_{12}+L-c.F_{12}.L,$$

where

$$c=\frac{H_{12}'P}{n_3}=\frac{0.0057285}{1.336}.$$

Also,

$$\frac{H_{12}H}{n_1}=\frac{c.L}{F} \qquad \frac{P'H'}{n_7}=-\frac{c.F_{12}}{F},$$

where H, H′ designate the positions of the principal points of the eye. Thus, we find:

$$F=+58.64 \text{ dptr.};$$

$$A_1H=+1.348 \text{ mm.}, \qquad A_1H'=+1.602 \text{ mm.}$$

If the focal lengths of the eye are denoted by f and f', then, since $f=n_1/F$ and $f'=-n_7/F$, we obtain:

$$f=+17.055 \text{ mm.}, \qquad f'=-22.785 \text{ mm.}$$

The focal points F, F′ are located as follows:

$$A_1F=-15.707 \text{ mm.}, \qquad A_1F'=+24.387 \text{ mm.}$$

In GULLSTRAND'S schematic eye the length of the eyeball is taken as 24 mm., and therefore the second focal point F′ is not on the retina but 0.387 mm. beyond it; so that the schematic eye is not emmetropic but hypermetropic (see § 153) to the extent of 1 dptr.

131. Combination of Three Optical Systems.—It is frequently the case, especially in problems connected with physiological optics, that we desire to find the resultant of three co-axial optical systems of known refracting powers F_1, F_2 and F_3 separated by given intervals c_1, c_2, where

$$c_1=\frac{H_1'H_2}{n_2}, \qquad c_2=\frac{H_2'H_3}{n_3},$$

the principal points of the component systems being designated by H_1, H_1'; H_2, H_2'; and H_3, H_3'. The indices of refraction of the first and last media of system I are denoted by n_1, n_2; of system II by n_2, n_3; and of system III by n_3, n_4.

Here let us employ the symbol D to denote the refracting power of the compound system (I+II), and the letters G, G′ to designate the positions of the principal points of this partial combination. Evidently, according to the formulæ derived in § 126, we may write:

$$D=F_1+F_2-c_1.F_1.F_2;$$

$$\frac{H_1G}{n_1}=\frac{c_1.F_2}{D}, \qquad \frac{H_2'G'}{n_3}=-\frac{c_1.F_1}{D}.$$

Now let F denote the refracting power of the combination of systems I, II and III, and let H, H′ designate the positions of the principal points of this compound system. Then if the reduced interval between (I+II) and III is denoted by k, that is, if

$$k = \frac{G'H_3}{n_3},$$

then also:

$$F = D + F_3 - k.D.F_3,$$

$$\frac{GH}{n_1} = \frac{k.F_3}{F}, \qquad \frac{H_3'H'}{n_4} = -\frac{k.D}{F}.$$

Since

$$\frac{G'H_3}{n_3} = \frac{G'H_2'}{n_3} + \frac{H_2'H_3}{n_3},$$

we find:

$$k = \frac{c_1.F_1 + c_2.D}{D}.$$

If now these equations are combined so as to eliminate D and k, the following *system of formulæ for the combination of three optical systems* will be obtained finally:

$$F = F_1(1 - c_2.F_2) + F_2(1 - c_1.F_1)\ (1 - c_2F_2) + F_3(1 - c_1.F_1);$$

$$\frac{H_1H}{n_1} = \frac{c_1}{1 - c_1.F_1} + \frac{c_2.F_3 - c_1.F_1}{F(1 - c_1.F_1)};$$

$$\frac{H_4'H'}{n_4} = -\frac{c_2}{1 - c_2.F_2} + \frac{c_2.F_3 - c_1.F_1}{F(1 - c_2.F_4)}.$$

In the special case when the *compound system is symmetrical with respect to system* II, that is, when $n_3 = n_2$ and $n_4 = n_1$ and $c_2 = c_1 = c$ and $F_3 = F_1$, the formulæ above will be simplified as follows:

$$F = (1 - c.F_1)\ (2F_1 + F_2 - c.F_1.F_2),$$

$$\frac{H_1H}{n_1} = \frac{H'H_3'}{n_1} = \frac{c}{1 - c.F_1}.$$

Thus, if an optical system is symmetrical with respect to a middle component part of the system, the principal points (H, H′) will be symmetrically placed, and their positions will be independent of the refracting power F_2 of the

middle system. These latter formulæ should be compared with the formulæ for a "thick mirror" to be developed in the following section.

132. "Thick Mirror."—The general formulæ which have been derived in this chapter are applicable also when the centered system of spherical surfaces includes one or more reflecting surfaces, provided that reflection is treated as a special case of refraction, according to the method explained in § 75. Thus, for example, if the rays are reflected at the kth surface of the system, we must put $n_{k+1} = -n_k$; and, consequently, if the reflecting power of this surface is denoted by F_k, we shall have $F_k = n_k/f_k = n_k/f'_k$, in accordance with the characteristic requirement that the focal lengths of a spherical mirror are identical, that is, $f = f'$ (see § 77).

A special case of much interest and practical importance occurs when *the last surface of the system acts as a mirror*, the rays of light arriving there being reflected back through the system as so to emerge finally at the first surface into the medium of index n_1 where they originated. For example, this happens always in the case of an ordinary glass mirror which is silvered at the back. The rays return into the air in front of a mirror of this kind after having twice traversed the thickness of the glass, and the failure to take account of the refractions from air to glass and from glass to air is sometimes responsible for serious errors in the measurement of the focal length of a glass mirror silvered at the back. The image produced by rays which have been partially reflected from the second surface of an ordinary lens is often very disturbing, although the intensity of the reflected light is usually comparatively feeble unless the second surface of the lens has been silvered.

The name "*thick mirror*" has been applied by Dr. SEARLE* to any combination of centered spherical refracting surfaces

* G. F. C. SEARLE: The determination of the focal length of a thick mirror. *Proc. Cambr. Phil. Soc.*, xviii, Part iii, 1915, 115–126.

wherein the rays are supposed to be reflected at the last surface and to return through the system in the opposite sense. It may easily be shown that a "thick mirror" as thus defined acts exactly like a single spherical reflecting surface (or "thin mirror," as we may call it, having in mind a certain analogy which exists here between lenses and mirrors), whose vertex and center have perfectly definite and calculable positions depending on the constants of the "thick mirror." This is proved by Dr. SEARLE in a simple manner as follows:

In Fig. 173 the system is represented as consisting of three spherical surfaces, the first two forming a thick lens

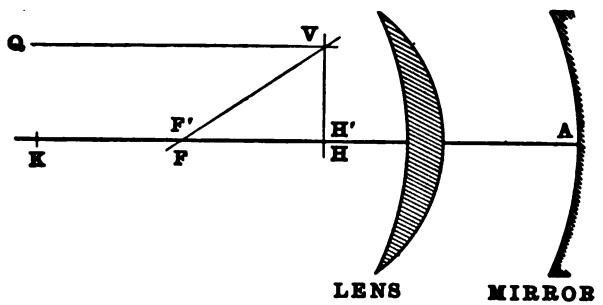


FIG. 173.—Diagram of "thick mirror" system.

and the last surface being a spherical mirror with its vertex at a point A on the axis of the lens. Draw the straight line QV parallel to the axis of the system to represent the path of an incident ray; which after traversing the lens and being reflected at the mirror will again emerge from the lens and cross the axis at the secondary focal point (F′) of the system. The point V designates the point of intersection of the incident ray QV and the corresponding emergent ray VF′, and hence this point must lie in the secondary principal plane of the system (§ 119). Consequently, the foot of the perpendicular let fall from V on to the axis will be the secondary principal point H′. But by the principle of the re-

versibility of the light-path (§ 29), if the straight line F'V is regarded as an incident ray, then VQ will be the path of the corresponding emergent ray, and since in this case the emergent ray is parallel to the axis, the corresponding incident ray F'V must cross the axis at the primary focal point F, so that the two focal points F and F' will be coincident. Moreover, the point V must lie in the primary principal plane, and hence the two principal planes are coincident. But these are the characteristics of a spherical mirror, and it is evident that the "thick mirror" is equivalent to a "thin mirror" with its vertex at H (or H') and its center at a point K such that HK=2HF.

The four images of PURKINJE are the catoptric images formed in the eye by reflection at the anterior and posterior surfaces of the cornea and the crystalline lens; which are of fundamental importance in determining the curvatures and positions of the refracting surfaces in the optical system of the eye. The first image is produced by direct reflection at the anterior surface of the cornea, but the optical systems which give rise to the three other images are more or less complicated. However, according to the above explanation, each of these systems may be reduced to a single reflecting surface of appropriate radius with its center at a certain definite place to be ascertained by the conditions of the problem. One of these cases will be investigated presently, as soon as the formulæ for a thick mirror have been developed.

The radius and positions of the vertex and center of the equivalent "thin mirror" may easily be calculated by means of the general formulæ which were obtained in the preceding section for a combination of three optical systems. Here the first system (I) of refracting power F_1 may be regarded as composed of the entire lens-system lying in front of the reflecting surface; while the mirror itself of reflecting power F_2 may be regarded as the second system (II). In this case the third system (III) will be the lens-system reversed, and

its refracting power will be the same as that of system I, that is, $F_3 = F_2$; but the principal points H_3 and H_3' of system III will coincide with the principal points H_1' and H_1, respectively, of system I. Above all we must impose here the conditions that

$$n_3 = -n_2 = -n', \quad n_4 = -n_1 = -n,$$

where n denotes the index of refraction of the medium of the object-space and n' denotes the index of refraction of the medium in contact with the reflecting surface. These conditions take account of the fundamental fact that the sense of propagation of the light is reversed by the mirror. The principal points H_2, H_2' of the mirror coincide with each other at its vertex which will be designated here by the letter A'. If therefore c_1, c_2 denote the reduced intervals between the first system and the mirror and between the mirror and the third system, we have:

$$c_1 = \frac{H_1'A'}{n'}, \quad c_2 = \frac{A'H_3}{n_3} = \frac{H_1'A'}{n'},$$

and hence

$$c_1 = c_2 = c, \text{ say.}$$

Moreover, if the radius of the reflecting surface is denoted by r', then $F = -\frac{2n'}{r'}$. Introducing these relations in the general formulæ for the combination of three optical systems (§ 131), we obtain the following expressions for finding *the reflecting power* (F_{13}) *and the positions of the principal points* H_{13}, H_{13}' *of a "thick mirror":*

$$F_{13} = (1 - c.F_1)\,(2F_1 + F_2 - c.F_1.F_2)$$

$$= (1 - c.F_1)\left\{2F_1 - \frac{2n'}{r'}(1 - c.F_1)\right\};$$

$$\frac{H_1H_{13}}{n} = \frac{H_1H_{13}'}{n} = \frac{c}{1 - c.F_1}.$$

Accordingly, we see not only that the principal points of a "thick mirror" are coincident with each other, but that the position of the vertex H_{13} of the equivalent "thin mirror" is entirely independent of the power F_2 or the curvature of

the actual mirror. The position of H_{13} does depend on the position of the vertex A′ of the actual mirror; but for any mirror placed at A′ the vertex of the equivalent "thin mirror" will be at the same point H_{13}. It may be noted that the formula for the reflecting power of a "thick mirror" is identical in form with the expression for the refracting power of a compound system which is symmetric with respect to a middle member (see end of § 131).

If the center of the equivalent "thin mirror" is designated by K, then its radius will be

$$H_{13}K = -\frac{2n}{F_{13}},$$

and hence

$$\frac{H_1K}{n} = \frac{c.F_2 - 2}{2F_1 + F_2 - c.F_1.F_2}.$$

If the surface of the mirror (II) is *plane*, then $F_2 = 0$, and in this case the formulæ for the equivalent "thin mirror" become:

$$F_{13} = 2F_1(1 - c.F_1), \quad \frac{H_1H_{13}}{n} = \frac{H_1H_{13}'}{n} = \frac{c}{1 - c.F_1}, \quad \frac{H_1K}{n} = -\frac{1}{F_1}.$$

The distinguishing characteristic of the imagery in a spherical mirror is that a pair of conjugate axial points M, M′ is harmonically separated by the vertex H and the center K of the mirror, that is, $(KHMM') = -1$ (§ 68). An interesting special case occurs when *one of the points* K *or* H *is at infinity;* for in that case the reflecting power of the mirror vanishes ($F = 0$). When the center of the mirror is at an infinite distance from it, the mirror lies midway between object and image ($MH = HM'$) and the lateral magnification is equal to $+1$ ($y' = y$); which is the case of an ordinary plane mirror. But, on the other hand, if the mirror itself is at an infinite distance, while the center K remains in the region of finite space, it is the center of the mirror in this case that is always midway between object and image, that is, $MK = KM'$, and now the lateral magnification will be equal to -1, that is, the image will be of the same size as

the object but inverted ($y' = -y$). Both of these special cases may be realized by a "thick mirror"; for the condition that the reflecting power of the equivalent "thin mirror" shall vanish ($F_{12} = 0$) requires that either

$$2F_1 + F_2 - c.F_1.F_2 = 0,$$

or

$$1 - c.F_1 = 0.$$

In the former case the center of the mirror (K) is at infinity, and in the latter case the vertex of the mirror (H_{12}) is at infinity. If therefore the distance between the anterior lens-system and the final reflecting surface of a "thick mirror" is $c = 1/F_1$, the system will produce an inverted image of the same size as the object, no matter where the object is placed.

As an illustration of the use of the formulæ for a "thick mirror," consider the optical system in the eye which produces the third of the so-called PURKINJE images, to which allusion was made earlier in this section. The third image is formed by rays which coming from an external source enter the eye, and after having traversed the cornea system and the aqueous humor are reflected at the anterior surface of the crystalline lens; whence returning through the same media in the reverse order they issue again into the air. In order to find the "thin mirror" which is equivalent to this system, we shall employ the constants of GULLSTRAND's schematic eye as given in § 130. The vertex of the anterior surface of the cornea will be designated by A_1 and the principal points of the cornea-system by H_1 and H_1'. We found that $A_1H_1 = -0.0496$ mm. and $A_1H_1' = -0.0506$ mm.; also, $F_1 = +43.05$ dptr., where F_1 denotes the refracting power of the cornea-system. The reflecting power of the anterior surface of the lens is given by the formula:

$$F_2 = -\frac{2n_3}{r_3},$$

where $n_3 = 1.336$ and $r_3 = +0.010$ m.; accordingly, we find:

$$F_2 = -267.2 \text{ dptr.}$$

The reduced distance between the cornea-system and the first surface of the lens is:

$$c = \frac{H_1'A_3}{n_3},$$

where A_3 designates the vertex of this surface; $A_1A_3 = 0.00036$ m. Thus, we obtain:

$$c = 0.0027325.$$

Substituting these numerical values in the system of formulæ for a "thick mirror," we find for the reflecting power of the equivalent "thin mirror" in this case:

$$F_{13} = -132.062 \text{ dptr.};$$

and for the positions of its vertex H_{13} and its center K:

$$H_1H_{13} = +3.0968 \text{ mm.}, \quad H_1K = +18.2412 \text{ mm.}$$

Accordingly, the system that produces the third of the PURKINJE images in GULLSTRAND's passive schematic eye is equivalent to a convex mirror of radius 15.14 mm. with its vertex at a distance of 3.047 mm. from the vertex of the anterior surface of the cornea.

Formulæ for calculating the reflecting power F_{13} of a "thick mirror" may also be obtained in terms of different data from those employed in the expressions which have been deduced above. Suppose, for example, that we are given the refracting power (F) of a centered system of spherical refracting surfaces, the positions of the principal points of the system (H, H'), and the indices of refraction of the first and last media (n, n'); together with the radius (r') and the position of the vertex (A') of the last surface; and that it is required to determine in terms of these data the characteristics of the imagery produced by light which proceeding from the object-space through the system is partially reflected at the last surface and again partially refracted at the first surface into the original medium. In order to solve this problem in the simplest way, it is convenient to employ a mathematical artifice which will be found to be serviceable in other optical problems. The refracting power of an infinitely thin concentric lens is equal to zero, and it is easy to

show that such a lens may be inserted anywhere in an optical system without affecting at all the resultant imagery (see § 90). Let us suppose, therefore, that the given optical system is terminated by an infinitely thin layer of material of index n', bounded by two concentric spherical surfaces, the first of which coincides with the last surface of the given system. Under these circumstances the resultant system may be considered as compounded of three component systems, namely, (1) the given system of refracting power $F_1 = F$, (2) a mirror of reflecting power $F_2 = -\frac{2n'}{r'}$, and (3) the given system reversed ($F_3 = F$). Hence, if

$$c = \frac{\mathrm{H'A'}}{n'},$$

the following formulæ will be obtained in the same way as above:

$$F_{13} = (1 - c.F)\left\{2F - \frac{2n'}{r'}(1 - c.F)\right\},$$

$$\frac{\mathrm{HH_{13}}}{n} = \frac{\mathrm{HH_{13}'}}{n} = \frac{c}{1 - c.F};$$

which are similar in form to the previous expressions, but c here has a different meaning and F denotes the refracting power of the entire lens-system and not merely of that part of the system which is in front of the reflecting surface.

A problem of considerable interest, especially in connection with the optical system of the human eye, is the investigation of the procedure of the light which after being partially reflected at the last surface of the system (as in the case above) is also partially reflected at the first surface, so that it emerges finally into the last medium of index n'. The imagery in this case may be determined by adding a second infinitely thin concentric lens, which is assumed to be made of material of index n and whose second surface coincides with that of the first surface of the system. Accordingly, now we shall have five systems in all, namely, the first three systems whose reflecting power F_{13} was obtained above,

a fourth system consisting of the first surface of the lens-system acting as a mirror, whose reflecting power is $F_4=\frac{2n}{r}$, where r denotes the radius of this surface, and a fifth system of refracting power $F_5=F$. The entire system, whose refracting power may be denoted by F_{15}, and whose principal points may be designated by H_{15}, H_{15}', may, therefore, be considered as compounded of 3 systems of powers F_{13}, $2n/r$ and F, separated by the intervals c_1 and c_2, where (if A designates the vertex of the first surface of the lens-system)

$$c_1=\frac{AH_{13}}{n}, \qquad c_2=\frac{AH}{n}.$$

Accordingly, by substituting F_{13} in place of F_1, $2n/r$ in place of F_2, and F in place of F_3 in the formulæ of § 131 for the combination of three optical systems, we obtain here:

$$F_{15}=F_{13}\,(1-c_2.F)+\frac{2n}{r}\,(1-c_1.F_{13})\,(1-c_2.F)+F\,(1-c_1.F_{13});$$

$$\frac{H_{13}H_{15}}{n}=\frac{c_1}{1-c_1.F_{13}}+\frac{c_2.F-c_1.F_{13}}{F_{15}\,(1-c_1.F_{13})};$$

$$\frac{H'H_{15}'}{n'}=-\frac{c_2}{1-c_2.F}+\frac{c_2.F-c_1.F_{13}}{F_{15}\,(1-c_2.F)}.$$

Being given the magnitudes denoted by n, n', r, r', and F and the positions of the points designated by A, A′ and H, H′, and having found by means of the previous formulæ the magnitude denoted by F_{13} and the position of the point designated by H_{13} (or H_{13}'), we can introduce these data and results in the expressions above and thus determine the refracting power F_{15} and the distances AH_{15}, $A'H_{15}'$ of the principal points H_{15}, H_{15}' from the vertices A, A′ of the first and last surfaces, respectively.

PROBLEMS

1. Find the refracting power and the positions of the focal points and principal points of each of the following glass lenses surrounded by air ($n=1$, $n'=1.5$); and make an ac-

curate sketch of each lens, marking the positions of the points mentioned.

(*a*) Double convex lens of radii 10 cm. and 15 cm. and of thickness 3 cm.

(*b*) Double concave lens with same data as above.

(*c*) Meniscus lens for which $r_1 = +5$ cm., $r_2 = +10$ cm., and $d = +3$ cm.

(*d*) Meniscus lens for which $r_1 = +6$ cm., $r_2 = +3$ cm., and $d = +2.52$ cm.

(*e*) A plano-convex lens with its curved surface, of radius 5 cm., turned towards the incident light; $d = +0.5$ cm.

(*f*) Symmetric convex lens, the radius of each surface being 5 cm.; $d = +0.5$ cm.

(*g*) Symmetric concave lens with same data as above.

(*h*) A meniscus lens with radii $r_1 = +5$ cm., $r_2 = +8$ cm., and thickness $d = +0.5$ cm.

(*i*) A meniscus lens with radii $r_1 = +8$ cm., $r_2 = +5$ cm., and thickness $d = +0.5$ cm.

(*j*) A meniscus lens with radii $r_1 = +8$ cm., $r_2 = +7$ cm., and thickness $d = +3$ cm.

(*k*) A plano-convex lens with its curved surface, of radius 5 cm., turned towards the incident light; $d = +5$ cm.

Answers:

	F in dptr.	A_1F in cm	A_2F' in cm.	A_1H in cm.	A_2H' in cm.
(*a*)	+ 8.000	−11.667	+11.250	+0.833	−1.250
(*b*)	− 8.667	+12.307	−12.692	+0.769	−1.154
(*c*)	+ 6.000	−18.333	+13.333	−1.667	−3.333
(*d*)	− 6.000	+21.333	−14.333	+4.667	+2.333
(*e*)	+10.000	−10.000	+ 9.667	0.000	−0.333
(*f*)	+19.666	− 4.915	+ 4.915	+0.170	−0.170
(*g*)	−20.335	+ 5.082	− 5.082	+0.164	−0.164
(*h*)	+ 3.959	−25.789	+24.421	−0.526	−0.842
(*i*)	− 3.542	+29.176	−27.647	+0.941	+0.588
(*j*)	0.000	∞	∞	∞	∞
(*k*)	+10.000	−10.000	+ 6.667	0.000	−3.333

2. In a symmetric lens ($r_1 = -r_2 = r$) surrounded by the

same medium (index n) on both sides, show that we have the following system of formulæ:

$$N=(n'-n)\{(n'-n)\,d-2n'.r\};$$

$$F=-\frac{N}{n'r^2};\ \frac{A_1F}{n}=-\frac{A_2F'}{n}=-\frac{r}{N}\left\{(n'-n)\,d-n'.r\right\};$$

$$\frac{A_1H}{n}=-\frac{A_2H'}{n}=-\frac{n'-n}{N}r.d.$$

3. If the first face of a lens is plane, and if the radius of the curved face is denoted by r, show that

$$F=-\frac{n'-n}{r};\quad \frac{A_1H}{n}=\frac{d}{n'};\quad A_2H'=0;$$

$$\frac{A_1F}{n}=\frac{r}{n'-n}+\frac{d}{n'};\quad \frac{A_2F'}{n}=-\frac{r}{n'-n}.$$

And if the second face of the lens is plane,

$$F=\frac{n'-n}{r};\ A_1H=0;\ \frac{A_2H'}{n}=-\frac{d}{n'};\ \text{etc.}$$

If either face of a lens is plane, the refracting power of the lens is equal to that of the curved surface and is entirely independent of the thickness of the lens; and, moreover, one of the principal points coincides with the vertex of the curved face.

4. If the radii r_1 and r_2 of the two surfaces of a lens are both positive, and if r_2 is greater than r_1, show that the lens is convergent, provided the lens-medium is more highly refracting than the surrounding medium.

5. A "lens of zero-curvature" is a crescent-shaped meniscus for which $r_2=r_1=r$. Show that such a lens is always convergent unless it is infinitely thin; and that this is the case whether the lens-medium is more or less highly refracting than the surrounding medium.

6. Show that a meniscus lens for which

$$r_1>r_2>0 \text{ and } n'>n$$

is divergent provided its thickness is less than

$$\frac{n'(r_1-r_2)}{n'-n}.$$

7. Show that in any meniscus lens surrounded by air at least one of the principal points must lie outside the lens.

8. A so-called *concentric lens* is one for which the centers of curvature of the two faces are coincident ($d=r_1-r_2$). It may be double convex or meniscus. Show that the refracting power of a concentric lens surrounded by the same medium on both sides is

$$F=\frac{n(n'-n)}{n'}\left(\frac{1}{r_1}-\frac{1}{r_2}\right),$$

and that the principal points coincide at the common center of the two surfaces.

9. Find the refracting power and the positions of the focal points and principal points of each of the following concentric glass lenses ($n'=1.5$) surrounded by air ($n=1$); and draw accurate sketch of each lens showing the positions of the points named:

(*a*) Double convex lens with radii $r_1=+10$ cm., $r_2=-2$ cm.

(*b*) Meniscus lens with radii $r_1=+5$ cm., $r_2=+2$ cm.

Ans. (*a*) $F=+20$ dptr., $A_1F=+5$ cm., $A_2F'=+3$ cm., $A_1H=+10$ cm., $A_2H'=-2$ cm.; (*b*) $F=-10$ dptr., $A_1F=+15$ cm., $A_2F'=-8$ cm., $A_1H=+5$ cm., $A_2H'=+2$ cm.

10. Find the focal length and the positions of the principal points of a concentric glass lens surrounded by air ($n=1$, $n'=1.5$), with radii $r_1=+8$ cm., $r_2=+5$ cm.

Ans. $f=-40$ cm., $A_1H=+8$ cm., $A_2H'=+5$ cm.

11. What is the refracting power of a concentric glass meniscus lens surrounded by air ($n=1$, $n'=1.5$), the radii being 5 cm. and 3 cm.? Ans. $F=-4.44$ dptr.

12. The radius of the second surface of a concentric glass lens surrounded by air ($n=1$, $n'=1.5$) is $+3$ cm., and its refracting power is -2 dptr. Determine its thickness.

Ans. 6.59 mm. If it were not too heavy, this would be a fairly good form of spectacle glass for a near-sighted person.

13. If the two principal points of a lens surrounded by the same medium on both sides coincide with each other at

a point midway between the two vertices, what is the form of the lens? Ans. A solid sphere.

14. The refracting power of a symmetric glass lens surrounded by air ($n=1$, $n'=1.5$) is $+10$ dptr., and its thickness is 0.5 cm. Determine the radius of the first surface.

Ans. $+9.916$ cm.

15. A solid sphere is a symmetric concentric lens. If the radius is denoted by r ($r=A_1C$), show that we have the following system of formulæ for a solid sphere surrounded by the same medium (n) on both sides:

$$F=\frac{2n(n'-n)}{n'.r}\,;\quad A_1H=H'A_2=r\,;\quad A_2F'=FA_1=\frac{(2n-n')r}{(2n'-n)}\,.$$

16. If the plane surface of a glass hemisphere, of index n' and surrounded on both sides by a medium of index n, is turned towards the incident light, and if r denotes the radius of the curved surface, show that

$$F=-\frac{n'-n}{r}\,,\quad A_1H=-\frac{n.r}{n'}\,,\quad A_2H'=0,\quad A_1F=\frac{n^2r}{n'(n'-n)}\,,$$

$$A_2F'=-\frac{n.r}{n'-n}\,.$$

17. An object is placed in front of the plane surface of a glass hemisphere, of index 1.5 and radius 3 inches, at a distance of 10 inches from this surface. Find the position, nature and size of the image.

Ans. A real, inverted image, of same size as object, will be formed at a distance of 25 inches from the object.

18. What is the refracting power of a glass sphere ($n'=1.5$), $16\frac{2}{3}$ cm. in diameter, (a) surrounded by air ($n=1$), and (b) surrounded by water ($n=\frac{4}{3}$)?

Ans. (a) $+8$ dptr.; (b) $+3\frac{5}{9}$ dptr.

19. The radius of each surface of a symmetric convex glass lens ($n'=1.5$) is 10 cm., and the thickness of the lens is 5 mm. What is its refracting power (a) when the thickness is neglected, and (b) when the thickness is taken into account? Ans. (a) $+10$ dptr.; (b) $+9\frac{11}{12}$ dptr.

20. The radii of a convex meniscus glass lens ($n'=1.5$)

surrounded by air ($n=1$) are 2.5 cm. and 5 cm. (*a*) If the lens is infinitely thin, what is its refracting power? (*b*) If the thickness of the lens is 1 cm., what is its refracting power?

Ans. (*a*) $F=+10$ dptr.; (*b*) $F=+11\frac{1}{3}$ dptr.

21. Determine the focal length (f) of a glass lens of index 1.5 surrounded by air for which $r_1=+10$, $r_2=+9$, (1) when thickness $d=0$, and (2) when thickness $d=+1$.

Ans. (1) $f=-180$; (2) $f=-270$.

22. A plane object is placed at right angles to the axis of a plano-convex lens at a distance of 8.77 cm. in front of its curved surface. The lens is made of glass of index 1.52, and the thickness of the lens is 0.5 cm. The radius of the curved surface is 4.56 cm. Show that the image will be at infinity, and that, in order to see distinctly the image of a point in the object which is 2 cm. from the axis, an eye behind the lens must look in a direction inclined to the axis of the lens at an angle of nearly 12° 51′.

23. The refracting power of a meniscus spectacle glass is +6 dptr., and $r_2=2r_1$, $d=6$ mm. The index of refraction is 1.5. Find the radii r_1 and r_2 and the vertex refraction V.

Ans. $r_1=+4.36$ cm., $r_2=+8.72$ cm., $V=+6.29$ dptr.

24. The thickness of a spectacle glass is 4.75 mm., and the index of refraction is 1.5. The refracting power of the first surface is +15.4 dptr., and that of the second surface is −9.1 dptr. Find the refracting power of the lens and its vertex refraction. Ans. $F=+6.74$ dptr.; $V=+7.09$ dptr.

25. A paraxial ray is incident on the cornea of GULLSTRAND's schematic eye (§ 130) in a direction parallel to the axis. Trace the path of this ray through the eye and determine the position of the secondary focal point F′ (see calculation-scheme, § 181); and calculate the focal lengths f, f' according to the formulæ derived in problem No. 17 at the end of Chapter X.

Ans. Distance of F′ from the vertex of the cornea is 24.387 mm.; $f=+17.055$ mm., $f'=-22.785$ mm.

26. The reduced thickness of a symmetric spectacle glass is denoted by c. If V denotes its vertex refraction, show that

$$F = \frac{V(\sqrt{4+c^2.V^2}-c.V)}{2}.$$

27. A hollow globe of glass is filled with water. The diameter of the water sphere is 8.5 inches and the thickness of the glass shell is 0.25 inch. Show that a narrow beam of parallel rays directed towards the center of the globe will be converged to a point 4.68 inches from the outside surface, the indices of refraction of glass and water being $\frac{3}{2}$ and $\frac{4}{3}$, respectively.

28. What is the focal length of a combination of two thin convex lenses, each of focal length f, placed at a distance apart equal to $2f/3$? Ans. $3f/4$.

29. An optical system is composed of two thin convex lenses of refracting powers $+10$ dptr. and $+6\frac{2}{3}$ dptr., the stronger lens being towards the incident light. Find the refracting power of the combination and the positions of the principal points and focal points when the distance between the lenses is: (*a*) 5 cm.; (*b*) 25 cm.; and (*c*) 40 cm.

Ans. (*a*) Convergent system: $F = 13\frac{1}{3}$ dptr.; $A_1H = +2.5$ cm.; $A_2H' = -3.75$ cm.; $A_1F = -5$ cm.; $A_2F' = +3.75$ cm.; (*b*) Telescopic system: $F = 0$; focal and principal points all at infinity; (*c*) Divergent system: $F = -10$ dptr.; $A_1H = -26\frac{2}{3}$ cm.; $A_2H' = +40$ cm.; $A_1F = -16\frac{2}{3}$ cm.; $A_2F' = +30$ cm.

30. An optical system is composed of two thin lenses, namely, a front concave lens of power -10 dptr. and a rear convex lens of power $+6\frac{2}{3}$ dptr. Find the refracting power of the combination and the positions of the focal points and principal points, when the interval between the lenses is: (*a*) 2.5 cm.; (*b*) 5 cm.; (*c*) 6.25 cm.; (*d*) 20 cm.

Ans. (*a*) Divergent system: $F = -1\frac{2}{3}$ dptr.; $A_1H = -10$ cm.; $A_2H' = -15$ cm.; $A_1F = +50$ cm.; $A_2F' = -75$ cm.; (*b*) Tele-

scopic system: $F=0$, focal and principal points all at infinity; (*c*) $F=+\frac{5}{8}$ dptr.; $A_1H=+50$ cm.; $A_2H'=+75$ cm.; $A_1F=-70$ cm.; $A_2F'=+195$ cm.; (*d*) Convergent system: $F=+10$ dptr.; $A_1H=+13\frac{1}{3}$ cm.; $A_2H'=+20$ cm.; $A_1F=+3\frac{1}{3}$ cm.; $A_2F'=+30$ cm.

31. Two thin convex lenses of focal lengths f_1 and f_2 are separated by an interval equal to $2f_2$. If $f_1=3f_2$, what is the focal length of the combination?

Ans. Convergent system of focal length $3f_2/2$.

32. Two lenses, one convex and the other concave, are separated by an interval $2a$. The convex lens is the front lens, and its focal length is a, while that of the concave lens is $-a$. Find the focal length of the combination and the positions of the principal points and focal points.

Ans. $f=a/2$; $A_1H=A_2H'=-a$; $A_1F=3A_2F'=-3a/2$.

33. Where are the principal planes of a system of two thin convex lenses of focal lengths 2 inches and 6 inches, separated by an interval of 4 inches?

Ans. The principal planes coincide with the focal planes of the stronger lens.

34. The objective of a compound microscope may be regarded as a thin convex lens of focal length 0.5. inch. The ocular may also be regarded as a thin convex lens of focal length 1 inch. The distance between the two lenses is 6 inches. Where must an object be placed in order that its image may be seen distinctly by a person whose distance of distinct vision is 8 inches?

Ans. $\frac{46}{53}$ inch in front of the objective.

35. The focal lengths of the objective and ocular of a compound microscope are 0.5 inch and 1 inch, respectively. If the distance of distinct vision is 12 inches, find the distance between the objective and ocular when the object viewed is 0.75 inch from the objective. Ans. 2.42 inches.

36. A thin convex lens, of focal length 5 inches, is placed midway between two thin concave lenses each of focal length 10 inches. The distance between the first lens and the second

is 5 inches. Find the focal length of the system and the positions of the principal points.

Ans. $f = +6\frac{2}{3}$ inches; the principal points are $3\frac{1}{3}$ inches from the outside lenses.

37. In the preceding problem, suppose that the two outside lenses are concave, everything else remaining the same.

Ans. $f = +6\frac{2}{3}$ inches. The principal points are on opposite sides of the middle lens and $1\frac{2}{3}$ inches from it.

38. A thin convex lens, of focal length 10 inches, is placed in front of a concave mirror of focal length 5 inches, the distance between them being 5 inches. The light traverses the lens, is reflected at the mirror, and again passes through the lens. Find the focal length of this so-called "thick mirror" and the positions of the principal points.

Ans. $f = +6\frac{2}{3}$ inches; the principal points coincide with each other at a point 5 inches behind the vertex of the mirror.

39. In the preceding problem, suppose that the lens is concave, everything else remaining the same.

Ans. $f = +6\frac{2}{3}$ inches; the principal points coincide with each other at a point between the lens and the mirror and $3\frac{2}{3}$ inches from the former.

40. In front of each of the systems described in Nos. 35, 36, 37 and 38, an object, one inch high, is placed at a distance of 5 inches from the first member of the system. Find the position, size and nature of the image in each case.

Ans. In No. 35: A real, inverted image, 2 inches beyond the third lens and 0.8 in. high. In No. 36: A real, inverted image, 30 inches beyond the third lens and 4 inches high. In No. 37: A real, inverted image, 2 inches in front of the lens and 0.8 in. high. In No. 38: A real, inverted image, 30 inches in front of the lens and 4 inches high.

41. The center of a concave mirror, of radius r, coincides with the optical center of a thin lens, of focal length f, and the axes of lens and mirror are in the same straight line. The light traverses the lens, is reflected at the mirror, and again

traverses the lens. Show that the system is equivalent to a thin mirror of radius $rf/(r+f)$, with its center at the same place as the center of the given mirror.

42. A centered system of lenses (I) is placed in front of a spherical mirror (II), and the whole constitutes a "thick mirror," as explained in § 132. Show that the vertex A and the center C of the actual mirror are the images of the vertex H and the center K, respectively, of the equivalent "thin mirror," which are produced by the lens-system I in the medium of index n_2 between systems I and II.

43. A "thick mirror" consists of a thin lens of focal length f_1 and a spherical mirror of focal length f_2 placed co-axially so that the focal point of the mirror coincides with the optical center A_1 of the thin lens. Show that the focal length of the equivalent "thin mirror" is

$$f=\frac{f_1^2 f_2}{f_1^2-f_2^2};$$

and that the positions of the vertex H and the center K are given by the following expressions:

$$A_1H=\frac{f_1 f_2}{f_1-f_2}, \qquad A_1K=-\frac{f_1 f_2}{f_1+f_2}.$$

Does it make any difference whether the lens is convex or concave?

44. At each of the focal points of a thin convex lens of focal length f_2 is placed a thin lens of focal length f_1. Find the focal length of the combination of the three lenses and the positions of the principal points. Does it make any difference whether the two equal outside lenses are convex or concave?

$$\text{Ans. } f=\frac{f_1^2 f_2}{f_1^2-f_2^2}; \quad A_1H=H'A_3=\frac{f_1 f_2}{f_1-f_2}.$$

45. A thin convex lens of focal length 10 cm. is placed in front of a plane mirror at a distance of 8 cm. from it. Find the radius of the equivalent "thin mirror" and the position of its vertex H.

Ans. The equivalent "thin mirror" is a concave mirror of radius 50 cm. with its vertex 32 cm. behind the plane mirror.

46. The axes of three thin convex lenses are all in the same straight line, the interval between the first and second lenses being one inch and the interval between the second and third lenses being half an inch. The focal lengths of the first, second and third lenses are $\frac{3}{2}$, $\frac{1}{6}$ and $\frac{3}{8}$ inch, respectively. A plane object is placed at right angles to the axis of the lens-system; show that an inverted image of the same size as the object will be formed in the plane of the object.

47. A plano-concave flint glass lens of index 1.618 is cemented to a double convex crown glass lens of index 1.523. The radii and thicknesses are as follows: $r_1 = \infty$, $r_2 = +50.419$ mm., $r_3 = -74.320$ mm.; $d_1 = +2.15$ mm., $d_2 = +4.65$ mm. Find the focal length of the combination and the positions of the principal points.

Ans. $f = +192.552$ mm.; distances of principal points from the plane surface, $+5.466$ and $+7.908$ mm.

48. A plano-concave flint glass lens of index 1.618 is cemented to a double convex crown glass lens of index 1.523. The radii and thicknesses are as follows: $r_1 = +22.00$ mm., $r_2 = -19.65$ mm., $r_3 = \infty$; $d_1 = +2.60$ mm., $d_2 = +2.00$ mm. Find the focal length of the combination and the positions of the principal points.

Ans. $f = +52.26$ mm.; distances of principal points from plane surface, -5.03 and -3.36 mm.

49. The radii and thickness of a symmetric double convex lens are 10 cm. and 1 cm., respectively. The lens is made of glass of index 1.5 and surrounded by air of index unity. A portion of the light which enters the lens will be reflected at the second surface and partially refracted at the first surface from glass back into the air. Find the radius, reflecting power and position of the vertex of the equivalent "thin mirror."

Ans. Concave mirror of radius -53.050 mm., reflecting

power +37.7 dptr., with its vertex +6.90 mm. from the vertex of the first face of the lens.

50. In the case of the lens in the preceding problem, assume that the light is reflected internally twice in succession and issues finally at the second face into the air. Find the refracting power and the positions of the principal points for the imagery produced by these rays.

Ans. Refracting power, +53.08 dptr.; distances of principal points from vertex of first surface of the lens, +10.94 and −0.94 mm.

51. In GULLSTRAND's schematic eye in its state of *maximum accommodation* the crystalline lens consists of an outer symmetric double convex lens of index $n_4 = n_6 = 1.386$ (see § 130), enclosing an inner symmetric double convex "core" lens of index $n_5 = 1.406$; the inner portion being symmetrically placed with respect to the surrounding outer part. The radii of the surfaces are as follows:

Outer portion: $r_3 = A_3C_3 = +5.3333$ mm. $= -r_6 = C_6A_6$;

Inner portion: $r_4 = A_4C_4 = +2.6550$ mm. $= -r_5 = C_6A_5$.

Moreover,

$$A_3A_4 = A_3C_5 = A_5A_6 = C_4A_6 = 0.6725 \text{ mm.};$$
$$A_4A_5 = C_5C_4 = A_4C_4 = C_5A_5 = 2.6550 \text{ mm.}$$

The entire lens is surrounded by a medium of index $n_3 = n_7 = 1.336$. Show (1) that the refracting power of the inner portion or "core" lens is $F_{45} = +14.959$ dptr., and that its principal points are 1.9905 mm. from the anterior and posterior surfaces. Moreover, employing the formulæ of § 131, show (2) that the refracting power of the entire lens in case of maximum accommodation is $F_{36} = +33.056$ dptr. and that $A_3H_{36} = H_{36}'A_6 = +1.9449$ mm.

52. Using the data of the preceding problem, find the refracting power (F) and the positions of the principal points (H, H') of GULLSTRAND's schematic eye in its state of maximum accommodation: being given, according to the results of § 130, that the refracting power of the cornea system is $F_{12} = 43.053$ dptr. and that $A_1H_{12} = -0.0496$ mm., $A_1H'_{12} =$

-0.0506 mm., and also that for maximum accommodation $A_1A_3 = +3.2$ mm.

Ans. $F = +70.575$ dptr.; $A_1H = +1.772$ mm.; $A_1H' = +2.086$ mm.

53. Two thin lenses of focal lengths f_1 and f_2 are placed on the same axis with the second focal point (F_1') of the first lens coincident with the first focal point (F_2) of the second lens, so as to form an afocal or telescopic system. Show that the lateral magnification is constant and equal to $-f_2/f_1$, and that the angular magnification is likewise constant and equal to the reciprocal of the lateral magnification.

54. If (as in HUYGENS's ocular) two thin lenses are placed on the same axis with their second focal points in coincidence, show that the second focal point of the combination is midway between this common focal point and the second lens, and that the deviation produced by the second lens is twice that produced by the first (assuming that the angles are small).

CHAPTER XII

APERTURE AND FIELD OF OPTICAL SYSTEM

133. Limitation of Ray-Bundles by Diaphragms or Stops. —The geometrical theory of optical imagery which has been developed in Chapter X was based on the assumption of *punctual correspondence* between object-space and image-space, whereby each point of the object is reproduced by one point, and by one point only, in the image; and on this hypothesis simple relations in the form of the so-called image-equations (§ 123) were obtained for determining the position and size of the image in terms of the focal lengths of the optical system. When we attempted to realize the imagery expressed by these equations, we were obliged to confine ourselves to the so-called paraxial rays comprised within the narrow cylindrical region immediately surrounding the axis of symmetry or optical axis of the centered system of spherical refracting or reflecting surfaces. Based on the same assumptions, certain rules were given for constructing the image-point Q′ corresponding to a given object-point Q. For example, a pair of straight lines was drawn through Q (Fig. 174), one parallel to the optical axis and meeting the second principal plane of the system in a point V′, and the other going through the primary focal point F and meeting the first principal plane in a point W. The required point Q′ was shown to lie at the point of intersection of the straight line V′Q′, drawn through the second focal point F′, with the straight line WQ′ drawn parallel to the axis. The position of the point Q′ having been located, the problem was considered as solved, and we were not particularly concerned with inquiring whether the straight lines used in the construction represented the paths of ac-

tual rays that formed the image at Q'. As a matter of fact, the pair of geometrical lines which is employed here will generally not belong to the bundle of optical rays by which the imagery is actually produced; and a glance at the diagram will show how the diameter of the lens and the size of

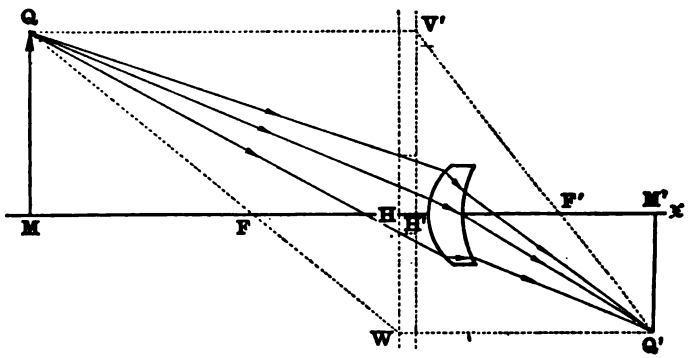


FIG. 174.—Effective rays as distinguished from rays used in making geometrical constructions.

the object control the selection of the rays that are really effective in producing the image.

In Chapter I attention was called to the fact that every optical instrument is provided with some means of cutting out such portions of a bundle of rays as for one reason or another are not desirable; which is usually accomplished, as has been explained, by interposing in the paths of the rays at some convenient place a plane opaque screen at right angles to the axis containing a circular aperture with its center on the axis. There may, indeed, be several such *diaphragms* or *stops* disposed at various places along the axis of the instrument. A perforated screen of this kind is called a *front stop*, a *rear stop* or an *interior stop*, according as it lies in front of, behind or within the system, respectively. The rims and fastenings of the lenses act in the same way as the diaphragms to limit the ray-bundles. The stops have various duties to perform, their chief functions being

to cut off the view of indistinct parts of the image (limitation of the field of view), to cut out such rays as would tend to mar the perfection of that part of the image which is to be inspected (limitation of the aperture of the system), and, finally, to nullify injurious reflections from the sides of the tube or other parts of the instrument.

134. The Aperture-Stop and the Pupils of the System.— To an eye looking into the instrument from the side of the object, a front stop (which may be the rim of the first lens of the system) will be the only one that will be visible directly. Any other stop or lens-rim will be seen only by means of the real or virtual image of it that is cast by that part of the optical system which is between it and the eye. Similarly, if the eye is directed towards the instrument from the image-side, an interior stop or a front stop may be seen by means of the image of it that is produced by the part of the system that lies between it and the eye. Now these impalpable stop-images, whether visible or not, are just as effective in cutting out the rays as if they were actual material stops; because, obviously, any ray that goes through an actual stop must necessarily pass either really or virtually through the corresponding point of the stop-image; whereas a ray that is obstructed by a stop will not go through the opening in the stop-image.

That one of the stops which by virtue of its size and position with respect to the radiating object is most effective in cutting out the rays is distinguished as the *aperture-stop* of the system (§ 11), and in order to determine which of the several stops performs this office, it is necessary, first of all, to assign the position of the axial object-point M, without which the aperture of the system can have no meaning. Accordingly, we must suppose that the instrument is focused on some selected point M on the axis, which is reproduced by an image at the conjugate point M′. The transversal planes at right angles to the axis at M and M′ will be a pair of conjugate planes, for it is assumed here that the imagery

is ideal and of the same character as that produced by paraxial rays. Now this pair of conjugate planes plays a very important *rôle* in the theory of an optical instrument, so that hereafter we shall refer to the object-plane as the *focus-plane* (or the plane which is in focus on the screen) and to the conjugate plane in the image-space as the *screen-plane*.

Now if the eye is supposed to be placed on the axis at the point M and directed towards the instrument, the stop or stop-image whose aperture subtends the smallest angle at M is called the *entrance-pupil* of the system. All the effective rays (§ 11) in the object-space must be directed towards points which lie within the circumference of the circular opening of the entrance-pupil. In general, the entrance-pupil is the image of the aperture-stop as seen by looking into the instrument in the direction of the light coming from the object; but if the aperture-stop is a front stop, it will also be the entrance-pupil.

On the other hand, when the eye is placed on the axis at the point M′ so as to look into the instrument through the other end, the stop or stop-image which subtends the smallest angle at M′ is called the *exit-pupil*, and all the effective rays when they emerge from the instrument must go, really or virtually, through the opening of the exit-pupil. In this statement it is tacitly assumed that M′ is a real image of M; otherwise, it would not be possible for the eye placed at M′ to look into the instrument through the end from which the rays emerge. But in any case the exit-pupil is the stop or stop-image which subtends the smallest angle at M′. Generally, the exit-pupil will be the image of the aperture-stop as seen by looking into the instrument from the image-side; but if the aperture-stop is a rear stop, it will be itself the exit-pupil.

Since the effective rays enter the system through the entrance-pupil in the object-space and leave it through the exit-pupil in the image-space, it is evident that *the exit-pupil is the image of the entrance-pupil, so that the pupil-*

centers, designated by O and O′, *are a pair of conjugate axial points with respect to the entire system.*

The apertures of the ray-bundles in the object-space are determined by the entrance-pupil of the system; and the exit-pupil has a similar office in the image-space. Each of the pupils is the common base of the cones of effective rays in the region to which it belongs.

135. Illustrations.—The name "pupil" applied to these apertures by ABBE was suggested by an analogy with the optical system of the eye. The pupil of the eye is the contractile aperture of the colored iris, the image of which produced by the cornea and the aqueous humor is the entrance-pupil of the eye corresponding to what is popularly called the "black of the eye," because it looks black on the dark background of the posterior chamber of the eye. Since the center O of the entrance-pupil is the image of the center K of the iris-opening formed by rays that are refracted from the aqueous humor through the cornea into the air, then, by the principle of the reversibility of the light-path, we may also regard K as the image of O formed by rays which are refracted from air ($n=1$) through the cornea into the aqueous humor ($n'=1.336$). The apparent place of the eye-pupil varies slightly in different individuals and in the same individual at different ages. If we assume that the point O is 3.03 mm. from the vertex (A) of the cornea, that is, if we put $u=0.00303$ m., then $U=n/u=330$ dptr. And if we take the refracting power of the cornea as $F=42$ dptr. (§ 130), then, since $U'=U+F$, we find $U'=372$ dptr. and consequently $u'=\mathrm{AK}=n'/U'=0.0036$ m.; so that with these data the plane of the iris is found to be at a distance of 3.6 mm. from the vertex of the cornea. Thus we see that the entrance-pupil of the eye is very nearly 0.6 mm. in front of the iris.

As a simple illustration of these principles, consider an optical system which consists of an infinitely thin convex lens, with a stop placed a little in front of it. In the diagram (Fig. 175) the straight line DG perpendicular to the

axis of the lens represents the diameter of the lens which lies in the meridian plane of the paper. The diameter of the stop-opening is shown by the straight line BC parallel to DG. The centers of the lens and stop are designated by

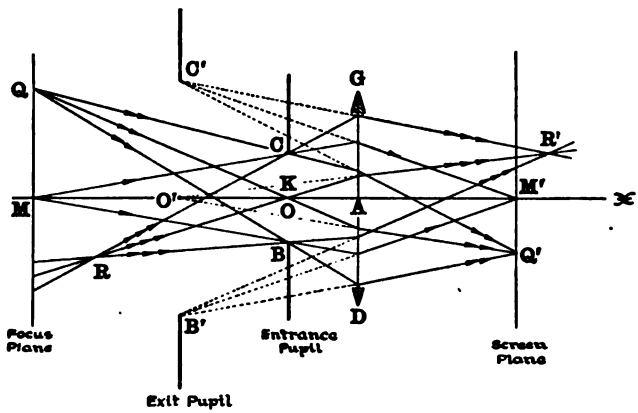


FIG. 175.—Optical system composed of thin convex lens with front stop.

A and K, respectively. The position of the focus-plane is determined by the axial object-point M, which in the figure is represented as lying in front of the lens beyond the primary focal plane. The solid angle subtended at M by the opening in the stop is supposed to be smaller than that subtended by the rim of the lens; that is, as here shown, $\angle AMC < \angle AMG$; and, consequently, the front stop acts here both as aperture-stop and entrance-pupil, so that the center K of the aperture-stop is likewise the center O of the entrance-pupil. Looking through the lens from the other side, one will see at O′ a virtual, erect image B′C′ of the aperture-stop BC, and hence this image is the exit-pupil of the system. The angle BMC is the aperture-angle of the cone of rays that come from the axial object-point M in the focus-plane; after passing through the system, these rays meet at M′ in the screen-plane, the aperture-angle of the bundle of rays

in the image-space being $\angle B'M'C'$. The effective rays coming from a point Q in the focus-plane are comprised within $\angle BQC$ in the object-space and $\angle B'Q'C'$ in the image-space. If the object-point does not lie in the focus-plane, and yet not too far from it, the opening BOC will act as entrance-pupil for this point also. Thus, for example, in order to construct the point R′ conjugate to an object-point R which does not lie exactly in the focus-plane, we have merely to draw the straight lines RB, RO, RC until they meet the lens, and connect these latter points with B′, O′, C′, respectively, by straight lines which will intersect in the image-point R′.

Again, consider a system composed of two equal thin convex lenses whose centers are at A_1 and A_2 (Fig. 176),

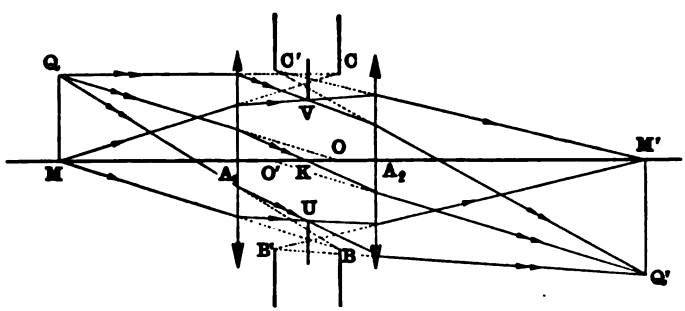


Fig. 176.—Optical system composed of two equal thin convex lenses with interior stop placed midway between the two lenses.

with a stop UV placed midway between them; if the center of the stop is designated by K, then $A_1K = KA_2$. The image of the stop as seen through the front lens is BOC, and its image as seen by looking through the other lens in the opposite direction is B′O′C′; these images being equal in size and symmetrically situated with respect to the stop itself. The image of the rim of each lens cast by the other lens should also be constructed, but for the sake of simplicity these images are not drawn in the figure, because the di-

ameters of the lenses are taken sufficiently large as compared with the diameter of the stop interposed between them at K to insure that the latter acts as aperture-stop with respect to the axial object-point M on which the instrument is supposed to be focused. Consequently, since the stop-image BC subtends at M an angle less than that subtended by the rim of the front lens or by the image of the rim of the second lens, it will be the entrance-pupil of the system; and, similarly, B′C′ which is the image of BC formed by the system as a whole will be the exit-pupil. Thus, in order to construct the image-point M′ conjugate to the axial object-point M, we have merely to draw the straight line MC and to determine the point where this line meets the first lens; and from the latter point draw a straight line through the point V in the edge of the stop to meet the second lens; and, finally, draw the straight line which joins this latter point with the point C′ in the edge of the exit-pupil; this line will cross the axis at the required point M′ in the screen-plane. Similarly, drawing from the object-point Q the three rays QB, QO and QC, we can continue the paths of these rays from the first lens to the second through the points U, K and V, respectively, in the stop-opening; and since the rays must issue from the second lens so as to go through B′, O′ and C′, respectively, in the exit-pupil, their common point of intersection in the image-space will be the point Q′ conjugate to Q. In the diagram the point Q is taken in the focus-plane; but the same construction will apply also to determine the position of an image-point R′ conjugate to an object-point R which does not lie in the focus-plane.

136. Aperture-Angle. Case of Two or More Entrance-Pupils.—The angle $OMC=\eta$ (Figs. 175 and 176) subtended at the axial object-point M by the radius OC of the entrance-pupil is called the *aperture-angle* of the optical system. If we put $OC=p$ (where p is to be reckoned positive or negative according as the point C lies above or below the axis)

and $OM = z$, then $\tan\eta = -p/z$. In like manner, if $\eta' = \angle O'M'C'$ denotes the angle subtended at the point M′ conjugate to M by the corresponding radius of the exit-pupil ($O'C' = p'$), and if also $O'M' = z'$, then $\tan\eta' = -p'/z'$.

The pupils of an optical system depend essentially, as has been stated, on the position of the axial object-point M on

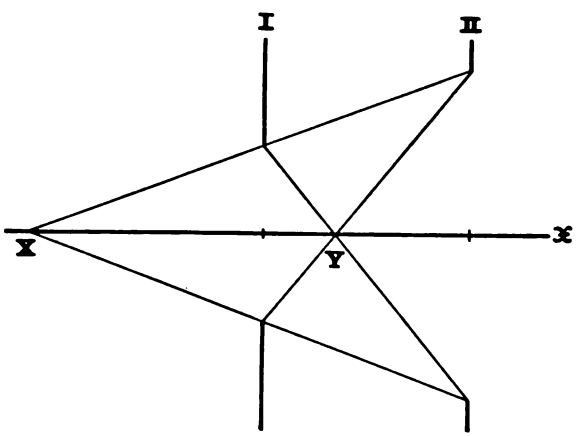


Fig. 177.—Case of two entrance-pupils.

which the instrument is focused. In the diagram (Fig. 177) I and II represent a pair of stops or stop-images as seen by an eye looking into the front end of the instrument. Join one end of the diameter of one of these openings by straight lines with both ends of the diameter of the other opening; and let the points where the straight lines cross the axis be designated by X and Y. The two apertures subtend equal angles at these points, and hence if the object-point M coincides with either X or Y, the entrance-pupil of the system may be either I or II; in fact, for these two special positions of M there will be two entrance-pupils, and, of course, also two exit-pupils. If the object-point M lies between X and Y, then in the case represented in the figure the opening II will subtend a smaller angle at M than the opening I so

that the former will act as the entrance-pupil. But for any other position of the axial object-point M besides those above mentioned the opening I will be the entrance-pupil.

137. Field of View.—The limitation of the apertures of the bundles of effective rays is not the only office of the stops and lens fastenings. One of their most important functions is to define the extent of the object that is to be reproduced in the instrument as has been pointed out in several simple illustrations in the earlier pages of this book (see §§ 9, 16, 73 and 98). In the adjoining diagram (Fig. 178), where the entrance-pupil of the system is represented by the opening BC, the other stops or stop-images in the object-space act like circular windows or port-holes through which the rays that are directed from the various parts of the object towards points in the open space of the entrance-pupil will have to pass if they are to succeed in getting through the instrument without being intercepted on the way. Evidently, that one of these openings which subtends the smallest angle at the center O of the entrance-pupil will limit the extent of the field of view in the object-space. This opening which is represented in the figure by GH is called the *entrance-port;* and the material stop or lens-rim which is responsible for it is called the *field-stop* (§ 9).

Let the straight line CH drawn through the upper extremities of the diameters BC and GH of the entrance-pupil and entrance-port meet the optical axis in the point designated by L and the focus-plane in the point designated by U. If this straight line is revolved around the axis of the instrument, the point U will describe a circle in the focus-plane around the axial object-point M as center; and it is obvious that any point in this plane within the circumference of this circle, or, indeed, any object-point contained inside the conical surface generated by the revolution of the straight line passing through C and H, may send rays to all parts of the entrance-pupil which will not be intercepted anywhere in the instrument. Thus, the entire aperture of the

entrance-pupil will be the common base of the cones of effective rays emanating from sources which lie within this region of the object-space.

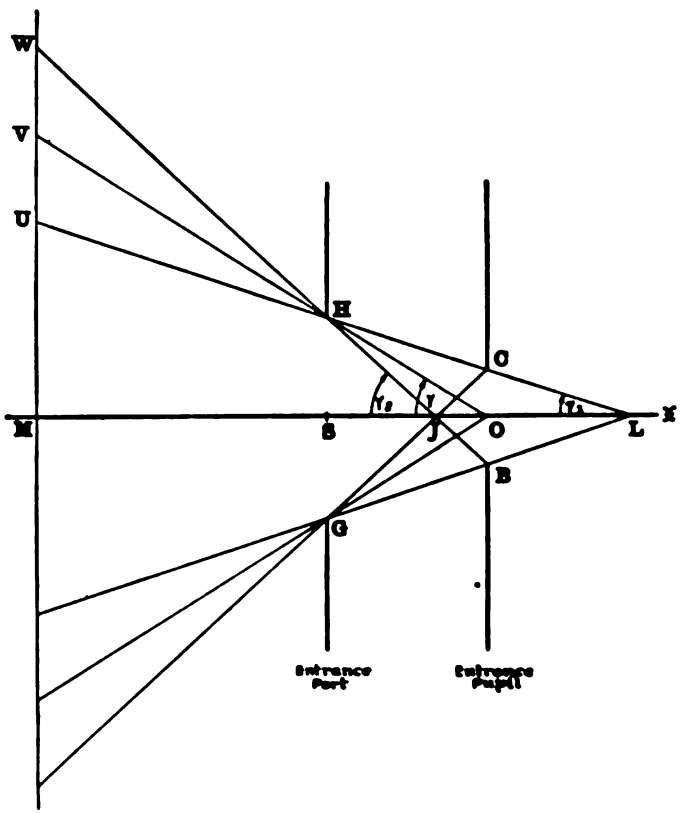


FIG. 178.—Field of view of optical system on side of object, determined by the entrance-pupil and the entrance-port.

Again, the straight line OH drawn through the center of the entrance-pupil and the upper edge of the entrance-port will determine a second limiting point V in the focus-plane which is farther from the optical axis than the first point U;

and in case of object-points lying in the focus-plane between U and V the sections of the bundles of effective rays made by the plane of the entrance-pupil will have areas that are comprised between the entire area of the opening of the entrance-pupil and half that area; and this will be true likewise with respect to all those points in the object-space that are contained between the two conical surfaces generated by the revolution of the straight lines CH and OH around the axis of symmetry. Such points will not lie outside the field of view, but although they can utilize more than half the opening of the entrance-pupil, they are not in a position to take advantage of the entire opening.

Finally, the straight line BH drawn through the lower edge of the entrance-pupil and the upper edge of the entrance-port, which crosses the optical axis at the point marked J, will determine an extreme point W in the focus-plane which is more remote from the axis than the point V; and it is evident from the figure that object-points in the focus-plane which lie in the annular space between the two circles described around M as center with radii MV and MW are even more unfavorably situated for sending rays into the entrance-pupil, because they cannot utilize as much as half of the pupil-opening. In fact, the effective rays which come from the farthest point W pass through the circumference of the pupil, and any point lying beyond W will be wholly invisible, that is, entirely outside the field of view of the instrument.

Thus, we see that the focus-plane is divided into zones by three concentric circles of radii MU, MV and MW. Object-points lying in the interior central zone send their light through the entrance-pupil without let or hindrance on the part of the field-stop; so that this is the brightest part of the field. But in the two outer zones there is a gradual fading away of light until we reach finally the border of complete darkness. The three regions of the field of view in the object-space are usually defined by the angles $2\gamma_1$, 2γ, and $2\gamma_2$

whose vertices are on the optical axis at the points L, O and J, respectively; so that $\gamma_1 = \angle SLH$, $\gamma = \angle SOH$ and $\gamma_2 = \angle SJH$. If the radii of the entrance-pupil and entrance-port are denoted by $p = OC$ and $b = SH$, and if the distance of the entrance-pupil from the entrance-port is denoted by $c = SO$, then

$$\tan\gamma_1 = -\frac{b-p}{c}, \quad \tan\gamma = -\frac{b}{c}, \quad \tan\gamma_2 = -\frac{b+p}{c}.$$

The field of view in the image-space is determined in like manner. The image of the entrance-port GH with its center at S, which is produced by the entire optical system, is the exit-port G'H' with its center at S'; and by priming all the letters in the expressions above a similar system of equations will be obtained for defining the three regions, $2\gamma_1'$, $2\gamma'$ and $2\gamma_2'$, of the field of view in the image-space. Generally, the edge of the field is considered as determined by the center of the pupil, that is, by the angle 2γ in the object-space and the angle $2\gamma'$ in the image-space.

138. Field of View of System Consisting of a Thin Lens and the Eye.—A simple but very instructive illustration of the principles explained in the foregoing section is afforded by an ordinary convex lens used as a magnifying glass. In order to obtain a virtual, magnified image with a lens of this kind, the distance of the glass from the object must not exceed the focal length of the lens, and then when the image is viewed through the glass, the iris of the observer's eye will act as the aperture-stop of the system, no matter where the eye is placed, provided the diameter of the pupil of the eye is less than that of the lens, as is practically nearly always the case. Moreover, since the pupil of the eye is the common base of the bundles of rays which come to it from the various parts of the image, it is the exit-pupil of the system, and its image in the glass is, therefore, the entrance-pupil. If the eye is placed on the axis of the lens between the lens and its second focal point (Fig. 179), the entrance-pupil will be a virtual image of the pupil of the eye and will

lie on the same side of the lens as the eye; if the eye is placed at the second focal point of the glass, the entrance-pupil will be at infinity (see § 144); and, finally, if, as represented in

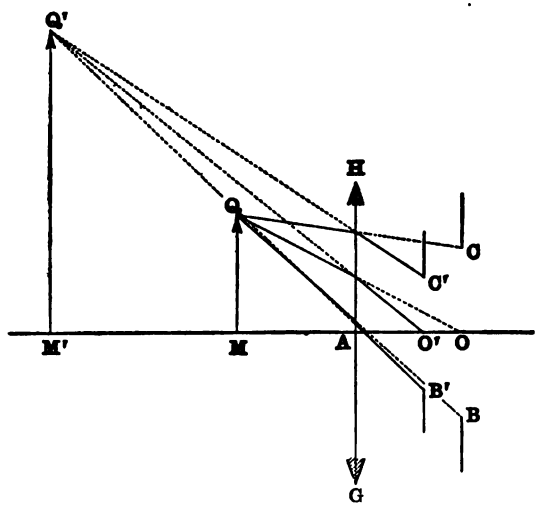


FIG. 179.—Field of view of thin convex lens when the eye is between the lens and its second focal plane.

Fig. 180, the eye is placed at a point O′ beyond the second focal point of the convex lens GH, the center of the entrance-pupil will be at a point O on the same side of the lens as the object MQ. The distance between the eye and the second focal point of a convex lens used as a magnifying glass is never very great, and, consequently, the distance of the center O of the entrance-pupil from the first focal point is relatively always quite large. The rim of the glass acts as the field-stop, and it is at the same time both the entrance-port and the exit-port of the system; and hence the field of view exposed to the eye in the image-space is entirely analogous to the field which would be seen by an eye looking through a circular window of the same form, dimensions and position as the lens. Since the exit-port is represented here as being

at a considerable distance from the exit-pupil, the field of view will appear vignetted, that is, the border will not be sharply outlined, but the field will fade out imperceptibly

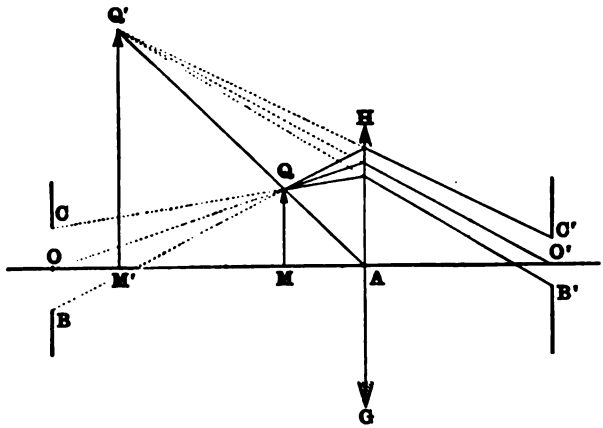


FIG. 180.—Field of view of thin convex lens when the eye is placed beyond the second focal plane.

towards the edges. If the diameter of the lens is denoted by $2b$, and if the distance of the eye from the lens is denoted by $c = AO'$, then $\tan\gamma' = -b/c$, where $\gamma' = \angle AO'H$. The extent of the field as measured by the angle $2\gamma'$ is independent of the size of the pupil of the eye. If the focus-plane coincides with the first focal plane of the magnifying glass, the diameter of the visible portion of the object will be $2y = -2f.\tan\gamma'$.

In a compound microscope or in an astronomical telescope the object-glass produces a real inverted image of the object, and this image is magnified by the ocular, which is essentially a convergent optical system on the order of a convex lens used as a magnifying glass. In the interior of the instrument between the object-glass and the ocular, at the place where the real image is cast by the object-glass, there is usually inserted a material stop, which cuts off the

"ragged edge" of the field of view, so that only the central portion which sends complete bundles of rays through the instrument is visible to the eye.

In the Dutch telescope the ocular is a divergent optical system which may be represented in a diagram by a con-

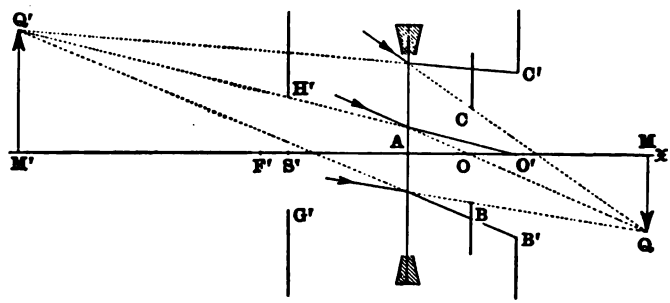


FIG. 181.—Ocular system of GALILEO's telescope represented in the diagram by a thin concave lens. Diagram shows how the rays, after having passed through the object-glass, enter the pupil of the observer's eye B'C'. Inverted image of distant object in the object-glass of the telescope is formed at MQ; M'Q' is the image of MQ in the ocular. G'H' is the image of the rim of the object-glass in the ocular. B'C' is the image of BC in the ocular.

cave lens (Fig. 181) which is placed between the object-glass and the real image of the object in the object-glass; so that so far as the ocular is concerned, this image is a virtual object, shown in the figure by the line-segment MQ. The eye in this case is usually adjusted very close to the concave lens. The pupil of the eye is represented in the figure by the opening B'C' with its center on the axis at O'; its image in the lens is BC. Here also, just as in the case of a convergent ocular, the pupil of the eye will act as the exit-pupil unless the diameter of the lens is so small that the lens-rim itself performs this office. The image of MQ is M'Q', which latter will be erect if MQ is inverted, and since MQ is always inverted in the simple telescope, the final image in the Dutch telescope is erect. In the case of the Dutch telescope the rim of the ocular lens does not limit

the field of view, but this is limited by the rim of the object-glass, which is the entrance-port of the telescope. Hence, the image of the object-glass in the ocular is the exit-port. This image (called the "eye-ring," § 159) is represented in the diagram by the opening G'H' with its center on the axis at S'. The object-point Q, as shown in the figure, is just at the edge of the field, because the image-ray coming from Q' which is directed towards the center O' of the exit-pupil is made to pass through the edge of the exit-port ($\gamma' = \angle S'O'H'$).

139. The Chief Rays.—Every bundle of effective rays emanating from a point of the object contains one ray which in a certain sense is the central or representative ray of the configuration and which may therefore be distinguished as the *chief ray* (see § 11). The ray which is entitled to this preëminence is evidently that one which in traversing the medium in which the aperture-stop lies passes through the center K of this stop. If the optical system is free from the so-called aberrations, both spherical and chromatic (as is assumed in the present discussion), the chief ray of the bundle may also be defined as that ray which in the object-space passes through the center O of the entrance-pupil; but the first definition is preferable because it is applicable to actual as well as to ideal optical systems.

The totality of the chief rays coming from all parts of the object constitute, therefore, a homocentric bundle of rays in the medium where the aperture-stop lies, and these rays proceed exactly as though they had originated from a luminous point at K.

If the aperture-stop is very narrow, comparable, say, with the dimensions of a pin-hole, the apertures of the bundles of effective rays will be correspondingly small; and in the limit when the opening in the stop may be regarded as reduced to a mere point at its center K, the ray-bundles will have collapsed into mere skeletons, so to speak, each one represented by its chief ray. It is because the chief rays are the last

survivors of the ray-bundles that it is particularly important in nearly all optical problems to investigate the procedure of these more or less characteristic rays.

140. The so-called "Blur-Circles" (or Circles of Diffusion) in the Screen-Plane.—Now if the cardinal points of the optical system are assigned, the image-relief corresponding to a three-dimensional object may be constructed point by point, according to the methods which have been explained. But, as a matter of fact, the image produced by

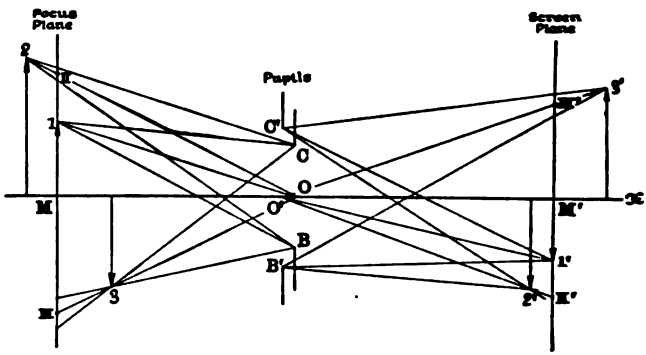


FIG. 182.—Diagram showing how object-relief and image-relief are projected in focus-plane and screen-plane from entrance-pupil and exit-pupil, respectively; and the "blur circles" in these planes.

an optical instrument, instead of being left, as it were, floating in space, is almost invariably received on a surface or screen of some kind, as, for example, the ground-glass plate of a photographic camera. In case the image is virtual, as in a microscope or telescope, it is intended to be viewed by the eye looking into the instrument, so that here also in the last analysis the image is projected on the surface of the retina of the observing eye. This receiving surface is called technically the "screen," which affords also an explanation of the name screen-plane (§ 134) as applied to the plane conjugate to the focus-plane.

In the diagram (Fig. 182) the screen-plane is placed at

right angles to the axis at the point marked M′ which is conjugate to the axial object-point M, so that this point is seen sharply focused on the screen. Evidently, however, the optical system cannot be in focus for all the different points of the object-relief at the same time, because the screen-plane is conjugate to only one transversal plane of the object-space, namely, the focus-plane perpendicular to the axis at M. Thus, for example, the reproduction of a solid object such as an extended view of a landscape on the ground-glass plate of a camera is not an image at all in the strict optical sense of the term, inasmuch as it is not conjugate to the entire object with respect to the photographic objective. Only such points of the object as lie in the focus-plane will be reproduced by sharp clear-cut image-points in the screen-plane (as, for example, the point marked 1 in the figure); whereas object-points situated to one side or the other of the focus-plane will be depicted more or less indistinctly on the screen-plane by small luminous areas which are sections cut out by this plane from the cones of image-rays emanating originally from points of the object such as those marked 2, 3 in the diagram. These little patches of light on the screen, which are usually elliptical in form, and whose dimensions depend on obvious geometrical factors, such as the diameter and position of the exit-pupil, etc., are the so-called circles of diffusion or "*blur-circles*," in consequence of which details of the image as projected on the screen are necessarily impaired to a greater or less degree.

It is a simple matter to reconstruct the object-figure which is optically conjugate to this configuration of image-points and "blur-circles" in the screen-plane, which will obviously be a similar configuration of object-points and "blur-circles" all lying in the focus-plane. Moreover, since the exit-pupil is conjugate to the entrance-pupil, the cones of rays in the object-space corresponding to those in the image-space may be easily constructed by taking the points

of the object-relief as vertices and the entrance-pupil as the common base of these cones. The *tout ensemble* of the sections of all these bundles of object-rays made by the focus-plane will evidently be the figure in the object-space that corresponds to the representation on the screen, and according to the theory of optical imagery these two plane configurations will be similar. This "vicarious" object in the focus-plane is sometimes called the *projected copy of the object-relief*, because it is obtained by projecting the points of the object from the entrance-pupil on the focus-plane.

141. The Pupil-Centers as Centers of Perspective of Object-Space and Image-Space.—It hardly needs to be pointed out that the "blur-circles" which arise from this

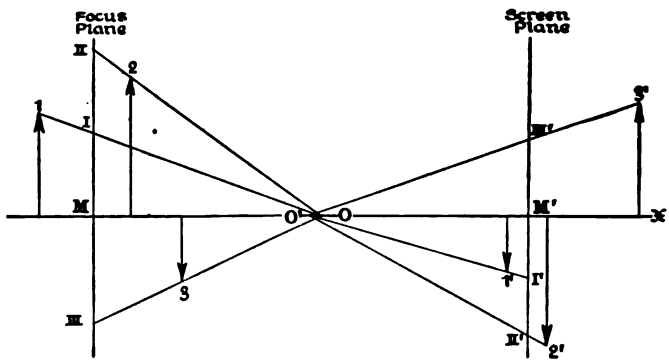


FIG. 183.—Projection of object-relief and image-relief in focus-plane and screen-plane from the centers of entrance-pupil and exit-pupil, respectively.

mode of reproducing a solid object on a plane (or curved) surface are due to no faults of the optical system itself, but are necessary consequences of the mode of representation, having their origin, in fact, in the object-space by virtue of the process employed. The only possible way of diminishing or eliminating the indistinctness or lack of detail in the reproduction of parts of the object that do not lie in the focus-plane consists in reducing the diameter of the aperture-stop, or in "stopping down" the instrument, as it is called.

If the stop-opening is contracted more and more until finally it is no larger than a fine pin-hole, the pupils likewise will tend to become mere points at their centers O, O′ (Fig. 183), and the "blur-circles" both in the focus-plane and in the screen-plane will diminish in area *pari passu* and ultimately collapse also into the points where the chief rays cross this pair of conjugate planes. The points marked I, II, III, etc., where the chief rays belonging to the object-points 1, 2, 3, etc., cross the focus-plane, and which are the centers of the so-called "blur-circles" in this plane, are obtained, therefore, by projecting all the points of the object from the center of the entrance-pupil on to the focus-plane. This mode of representing a three-dimensional object is, however, in no wise peculiar to the optical system itself, but is the old familiar process of *perspective reproduction* by central projection on a plane. Thus, the pupil-centers O, O′ are to be regarded as the *centers of perspective* of the object-space and image-space, respectively.

142. Proper Distance of Viewing a Photograph.—These principles explain why it is necessary to view a photograph at a certain distance from the eye in order to obtain a correct impression of the object which is depicted. Suppose, for example, that O, O′ (Fig. 184) designate the centers of the pupils of a photographic lens, and that an object NR is reproduced in the screen-plane by the perspective copy M′Q′ whose size is one kth of that of the projection MQ of the object in the focus-plane. Now if the picture is to produce the same impression as was produced by the original itself on an observer with his eye placed at O, the photograph must be held in front of the eye at a place P such that the visual angle KOP which it subtends at the center of rotation of the eye shall be equal to the angle QOM; that is, the distance PO in the figure must be equal to one kth of the distance of the center of the entrance-pupil from the focus-plane, or $PO = MO/k$. If (as is usually the case with a landscape lens) the focus-plane is at infinity, then PO will

be equal to the focal length (f) of the objective. Generally speaking, we may say, therefore, that the correct distance for viewing a photograph of a distant object is equal to the focal length of the objective, this distance being measured

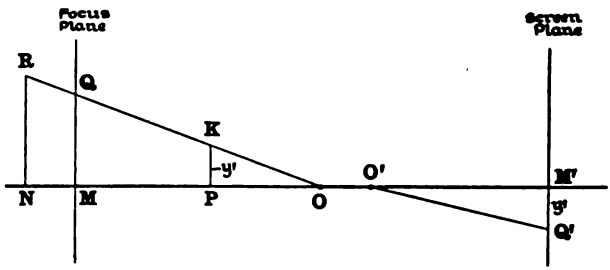


FIG. 184.—Correct distance of viewing photograph.

from the picture to the center of rotation of the observer's eye. Accordingly, if the focal length is less than the distance between the near point of the eye and the center of rotation, which in the case of a normal emmetropic eye of an adolescent is about 10 or 12 cm., it will be impossible to see the picture distinctly with the naked eye and at the same time under the correct visual angle. Moreover, even if the focal length of the photographic lens were not less than this least distance of distinct vision, the effort of accommodation which the eye has to make in order to focus the image sharply on the retina under the correct visual angle will superinduce an illusion which will be different from the impression of reality which it is the purpose of the picture to convey. In the case of a photograph made by an objective of very short focal length it is possible indeed to make an enlarged copy which may be viewed at the correct distance, but this is always more or less troublesome and expensive. Dr. VON ROHR has invented an instrument called a *verant* which is ingeniously designed to overcome as far as possible the difficulties above mentioned; so that viewed through this ap-

paratus the photograph is seen more or less exactly as the object appeared.

143. Perspective Elongation of Image.—If the screen-plane is not focused exactly on the image-point R′ (Fig. 185),

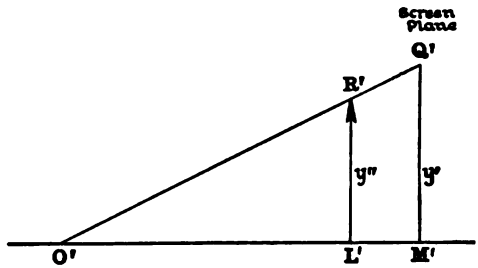


FIG. 185.—Perspective elongation of image.

this point will be shown on the screen by a "blur-circle" whose center will be at the point Q′ which is the projection of R′ from the center O′ of the exit-pupil. Let $e = L'M'$ denote the distance of the screen-plane M′Q′ from the image-plane L′R′, where L′, M′ designate the feet of the perpendiculars dropped from R′, Q′, respectively, on the axis. From the diagram we obtain the proportion:

$$\frac{M'Q'}{L'R'} = \frac{O'M'}{O'L'} = \frac{O'M'}{O'M' + M'L'};$$

which may be written:

$$y'/y'' = z'/(z' - e),$$

where $y' = M'Q'$, $y'' = L'R'$ and $z' = O'M'$. Moreover, since e may be regarded as small in comparison with z', we obtain:

$$y' - y'' = \frac{e}{z'}y'', \text{ approximately.}$$

The difference $(y' - y'')$ is the measure of the *perspective elongation* due to imperfect focusing.

If the exit-pupil is at infinity, then R′Q′ will be parallel to the axis and $y' = y''$; and under these circumstances, the perspective reproduction in the screen-plane will be of the same size as the image, no matter how much it is out of focus.

144. Telecentric Systems.—A common laboratory use of an optical instrument is to ascertain the size of an inaccessible or intangible object from the measured dimensions of its image as determined by means of a scale on which the

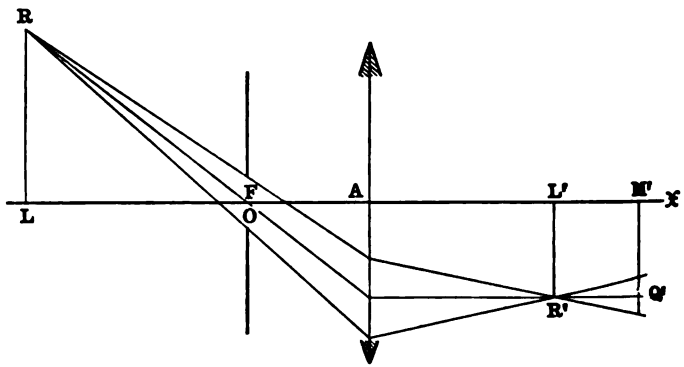


Fig. 186.—Telecentric optical system: Case of a thin convex lens with front stop in first focal plane. Object represented by LR; blurred image M'Q' appears of the same size as sharp image L'R'.

image is projected; but, in general, unless the scale is exactly in the same plane as the image, there will be a parallax error in the measurement of the image due to its perspective elongation. However, if the chief rays in the image-space are parallel to the axis, which may be effected by placing the aperture-stop so that the entrance-pupil lies in the primary focal plane of the instrument, as illustrated in Fig. 186, the perspective elongation vanishes ($y'-y''=0$, as explained in § 143); and, consequently, the image $y''=L'R'$ will appear of the same size as its projection $y'=M'Q'$, no matter whether it lies in the same plane as the scale or not.

Similarly, if the aperture-stop is placed so that the entrance-pupil is at infinity and the exit-pupil lies therefore in the secondary focal plane, the chief rays in the object-space will then all be parallel to the optical axis.

Systems of this description in which one or other of the two projection centers O, O′ is at infinity are said to be

telecentric. This is the principle of nearly all systems for micrometer measurements of optical images.

A simple illustration of a device of this kind that is telecentric on the side next the object is afforded by the ophthalmic instrument called a *keratometer*, which, as the name implies, is intended primarily to measure the diameter of the cornea or the apparent diameter of the eye-pupil. It is used also to measure the distance of a correction-glass (§ 154) from an ametropic eye (§ 153), which is an important factor in the prescription of spectacles. The instrument consists essentially of a long narrow tube, near the middle of which is mounted a convex lens of low power adjusted so that its second focal point F′ coincides exactly with the center of a small aperture in a metal disk placed at the end of the tube where the observer puts his eye. At the opposite end of the tube a scale graduated in half-millimeters is mounted so that

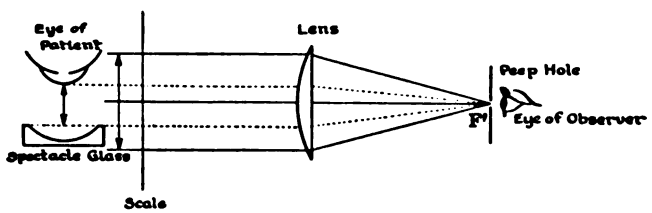


FIG. 187.—Diagram of instrument called keratometer, as used to measure the distance of spectacle glass from the cornea of the eye.

its upper edge coincides with a horizontal diameter of the tube at this place. The upper part of this end of the tube is cut away in order to admit sufficient light to illuminate the scale.

When the keratometer is used to measure the distance between the vertex of the cornea and the vertex of the correction-glass, it is placed with its axis at right-angles to the line of sight of the patient, as represented in the diagram (Fig. 187), the scale being brought as near as possible to

the patient between his eye and the spectacle-glass. The distance AB to be measured is projected on the scale by rays that are parallel to the axis of the lens, so that when the observer looks through the instrument he can read off this distance on the image of the scale.

Practically the same principle is employed also in BADAL's *optometer* for measuring the visual acuity of the eye. It

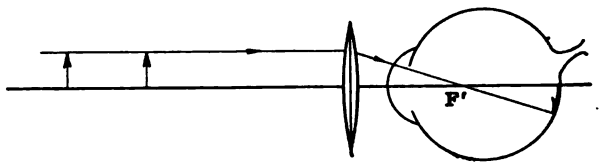


FIG. 188.—BADAL's optometer, with second focal point (F)' of convex lens at first nodal point of patient's eye; forming in conjunction with the eye a telecentric system.

consists of a single convex lens mounted at one end of a long graduated bar which is provided with a movable carrier holding a test-chart of some kind. If the lens, which usually has a refracting power of about 10 dioptries, is adjusted about 9 cm. in front of the cornea so that its second focal point F′ coincides with the nodal point of the eye (Fig. 188), a ray meeting the lens in a direction parallel to the axis will emerge from it so as to go through the nodal point of the

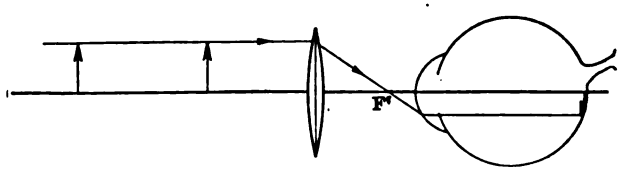


FIG. 189.—BADAL's optometer, with second focal point (F') of convex lens at first focal point of patient's eye; forming in conjunction with the eye a telescopic system.

eye and thence to the retina without change of direction. Accordingly, just as though a narrow aperture were placed

at the nodal point of the eye, the size of the retinal image will not be altered whether the object or chart on the bar be far or near; whereas the distinctness with which the details of the object are seen, which affords the measure of the visual acuity, will depend on the distance of the object.

Another method of using this optometer is to place the lens about 2 cm. farther from the eye, as shown in Fig. 189, so that now its second focal point lies in the anterior focal plane of the eye. Under these circumstances an incident ray proceeding parallel to the axis will emerge from the lens and cross the axis at the anterior focal point of the eye, so that after traversing the eye-media it will again be parallel to the axis. Consequently, here also the image formed on the retina will be of the same size no matter where the object is placed on the bar in front of the lens, just as if there were a narrow stop at the anterior focal point of the eye. In this latter adjustment the lens and the eye together constitute an optical system which is *telecentric on both* sides, that is, a *telescopic system* (§ 125).

PROBLEMS

1. A cylindrical tube, 2 cm. in diameter and 10 cm. long, is closed at one end by a thin convex lens of focal length 4 cm. If this end of the tube is pointed towards a distant object, what will be the position and diameter of the entrance-pupil? Ans. $6\frac{2}{3}$ cm. in front of the lens; diameter, $1\frac{1}{3}$ cm.

2. In the preceding problem, where would the object have to be in order that the lens itself might act as entrance-pupil?

Ans. In front of the lens, not more than 20 or less than 4 cm. away.

3. If in No. 1 the other end of the tube is closed by a thin eye-lens whose focal length is such that when the combination is pointed at an object 24 cm. from the object-glass, the bundles of rays issuing from the eye-lens are cylindrical, find

the positions of the pupils of the system and the focal length of the eye-lens.

Ans. Entrance-pupil $6\frac{2}{3}$ cm. in front of object-glass; exit-pupil coincides with eye-glass; focal length of eye-glass, 5.2 cm.

4. In the preceding problem what will be the answers on the supposition that the object is 12 cm. from the object-glass?

Ans. Entrance-pupil coincides with object-glass; exit-pupil is $6\frac{2}{3}$ cm. beyond eye-glass; focal length of eye-glass, 4 cm.

5. A real inverted image of an extended object is formed by the object-glass of a simple astronomical telescope in the primary focal plane of the eye-glass. The focal lengths of the object-glass and eye-glass are 2 feet and 1.5 inches, respectively, and their diameters are 6 inches and 1 inch, respectively. If the distance of the object from the object-glass is 240 feet, find the position and diameter of the entrance-port and the diameter of the portion of the object that is completely visible through the telescope.

Ans. Entrance-port is 30.21 feet from object-glass, and its diameter is 1.175 feet; diameter of visible portion of object, 5.865 feet.

6. A thin convex lens of focal length 10 cm. and diameter 4 cm. is used as a magnifying glass. If an eye adapted for parallel rays is placed at a distance of 5 cm. from the lens, what will be the diameter of the portion of the object that can be seen distinctly? Ans. 8 cm.

7. The diameter of a thin convex lens is 1 inch, and its focal length is 10 inches. The lens is placed midway between the eye and a plane object which is 10 inches from the eye. How much of the object is visible through the lens?

Ans. $1\frac{1}{3}$ inch.

CHAPTER XIII

OPTICAL SYSTEM OF THE EYE. MAGNIFYING POWER OF OPTICAL INSTRUMENTS

145. The Human Eye.—The organ of vision is composed of the eye-ball, wherein the visual impulses are produced by the impact of light; the optic nerve which transmits these excitations to the brain; and the visual center in the brain where the sensation of vision comes to consciousness.

The eye-ball (Fig. 190) lying in a bony socket on a cushion of fat and connective tissue, in which it is free to turn in all directions with little or no friction, consists of an almost spherical dark chamber, filled with transparent optical media which form the optical system of the eye (Fig. 191). The outer protecting envelope of the eye-ball is the tough, white membrane called, from its hardness, the *sclerotic coat* or *sclera*, popularly known as the "white of the eye." This opaque membrane is continued in front by a round opening or window called, on account of its horny texture, the *cornea*. The cornea is beautifully transparent, and its mirror-like surface forms a slight protuberance shaped something like a watch-glass or a prolate spheroid. In the interior of the eye the sclerotic coat is overlaid with the dark-colored *choroid* which contains the blood-vessels that nourish the eye and also a layer of brown pigment acting to protect the dark chamber of the eye from diffused light. Behind the cornea lies the *anterior chamber* filled with transparent fluid called the *aqueous humor*. This anterior chamber is limited behind the *iris*, which, rich in blood-vessels, imparts to the eye its characteristic color. This is an opaque screen or curtain which contains a central hole, the *pupil*, which is circular in the human eye. The aperture of a bundle of rays entering the

eye from a luminous point, in proportion to the dimensions of the eye, is enormous as compared, for example, with the same magnitude in a telescope; and the office of the pupil is

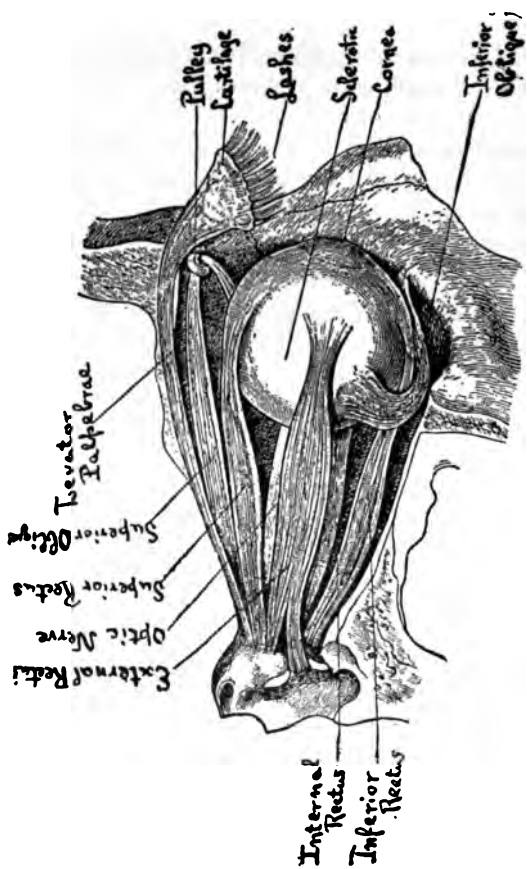


FIG. 190.—Diagram of the eye-ball of the human eye, showing the motor muscles.

to stop down this aperture to suitable proportions. The pupil contracts or dilates involuntarily and regulates the quantity of light that is admitted to the eye. In the structure of the iris there are two sets of fibers, the circular and

radiating; when the circular fibers contract, the pupil ;racts, and when the radiating fibers contract, the pupil

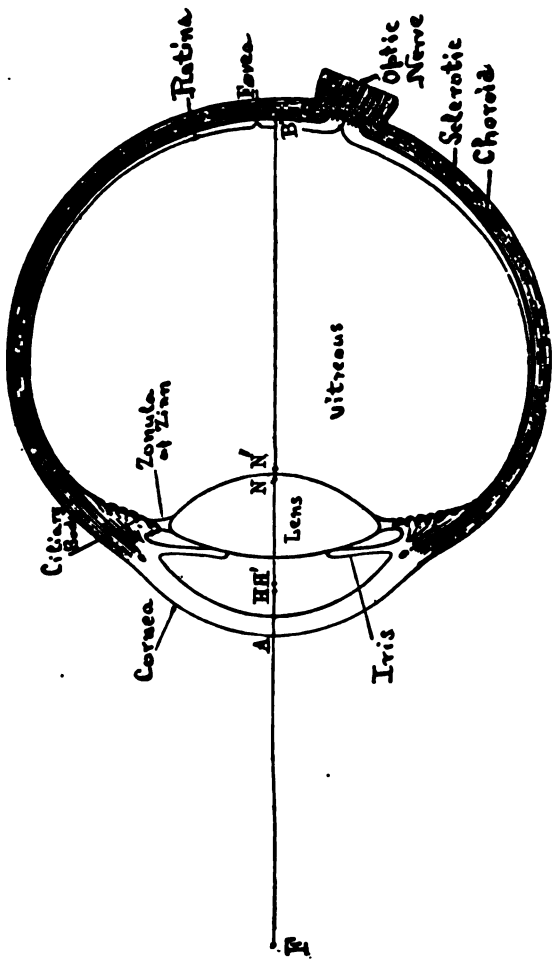


FIG. 191.—Diagram of the human eye, showing its principal features.

tes. In the front part of the eye the choroid lining is dered at the edge of the cornea by a kind of folded drapery so-called *ciliary body*, which is hidden from without be-

hind the iris and which contains the delicate system of muscles which control the mechanism of accommodation. The *crystalline lens* composed of a perfectly transparent substance is indirectly attached to the ciliary body by a band which surrounds the edge of the lens like a ring and which is disposed in radial folds somewhat after the manner of a neck-frill. This band is the *suspensory ligament* or *zonule of Zinn.* The lens itself is double convex, the posterior surface being more strongly curved than the anterior surface. The substance of the lens consists of layers of different indices of refraction increasing towards the center or core of the lens. The entire space behind the lens is filled with a transparent jelly-like substance called the *vitreous humor,* which has the same index of refraction as the aqueous humor, namely, 1.336.

The light-sensitive *retina* lying on the inside of the choroid is exceedingly delicate and transparent. In spite of its slight thickness which nowhere exceeds 0.4 mm., the structure of the retina is very complicated, and no less than ten layers have been distinguished (Fig. 192). The layer next the vitreous humor is composed of nerve-fibres spreading out radially from the optic nerve. This layer is connected with the following layer containing the large ganglion or nerve-cells, and this in turn is connected by an apparatus of fibers and cells with the peculiar light-sensitive elements of the retina, the so-called visual cells which form the "bacillary layer." These visual cells consist of characteristic elongated bodies which are distinguished as *rods* and *cones.* The rods are slender cylinders, while the cones or bulbs are somewhat thicker and flask-shaped. They are all disposed perpendicularly to the surface of the retina, closely packed together, so as to form a mosaic layer at the back of the retina.

Near the center of the retina at the back of the eye, a little to the temporal side, is located the *yellow spot* or *macula lutea,* where the visual cells are composed mostly of cones. This

is the most sensitive part of the retina, especially the minute pit or depression at the center of this area, called the *fovea centralis*, which consists entirely of cones densely crowded together.

As compared with an artificial optical instrument, the

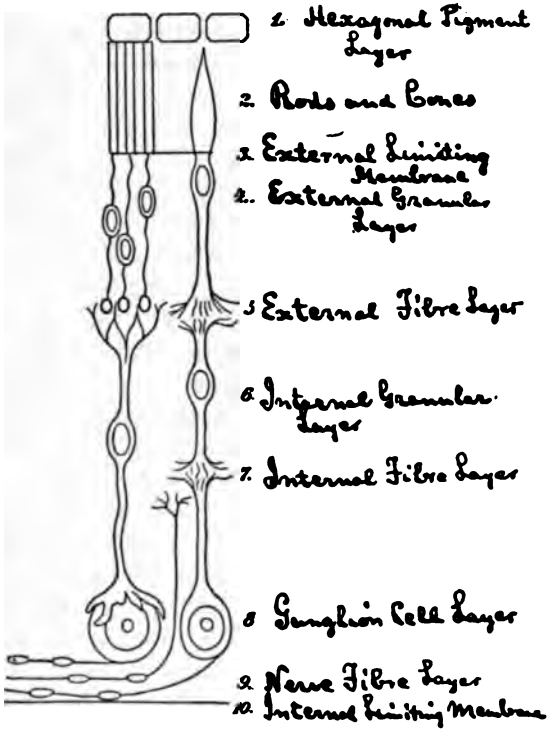

1. Hexagonal Pigment Layer
2. Rods and Cones
3. External Limiting Membrane
4. External Granular Layer
5 External Fibre Layer
6. Internal Granular Layer
7. Internal Fibre Layer
8 Ganglion Cell Layer
9. Nerve Fibre Layer
10. Internal Limiting Membrane

FIG. 192.—Structure of the retina of the human eye.

field of view of the immobile eye is very extensive, amounting to about 150° laterally and 120° vertically. The diameter of the *fovea centralis* corresponds in the field of vision of the eye to an angular space which may be covered by the nail of the fore finger extended at arm's length. In this part of the field vision is so acute that details of an object can be

distinguished as separate provided their angular distance is not less than one minute of arc (*cf.* §10). If the apparent size of an object is so small that its image formed on the retina at the *fovea centralis* covers only a single visual cell, the object ceases to have any apparent size at all and cannot be distinguished from a point. The size of the retinal image corresponding to an object whose apparent size is one minute of arc is found by calculation from the known optical constants of the eye to be 0.00487 mm. Anatomical measurements give a similar value for the diameter of a visual cell.

The inverted image cast on the retina of the eye has been compared to a sketch which is roughly outlined in the outer parts, but which is more and more finely executed in towards the center, until at the *fovea centralis* itself the details are exquisitely finished. Thus, only a comparatively small portion of an external object can be seen distinctly by the eye at any one moment. If all the parts of the field of view were portrayed with equal vividness at the same time and came to consciousness at once, the spectator would be completely bewildered and unable to concentrate his attention on a particular spot or phase of the object.

The ends of the rods next the choroid contain a coloring matter which is sensitive to light, the so-called *visual purple*, which is bleached white by exposure to bright light, but which is renewed in darkness by the layer of cells lying between the choroid and the retina. The light-disturbance arriving at the retina penetrates it as far as the bacillary layer of rods and cones, and the stimulus is transmitted back through the interposed apparatus to the layer of nerve-fibers and thence conducted to the *optic nerve* in communication with the brain.

Not far from the center of the retina, a little to the nasal side, the optic nerve pierces the eye-ball through the sclera and choroid. Here the retina is interrupted, so that any light which falls on the optic nerve itself cannot be perceived. This is the place of the so-called *blind-spot* (*punctum cæcum*)

of the eye. Corresponding to the area of the blind spot, there is a gap in the field of vision of the eye amounting to about 6° horizontally and 8° vertically. The dimensions of the blind spot are great enough to contain the retinal images of eleven full moons placed side by side. The optic nerve leaves the eyeball through a bony canal and passes thence to the visual center of the brain.

The mobility of the eye is produced by six muscles, the four recti and the two oblique muscles (Fig. 190). The recti originate in the posterior part of the socket and are attached by their tendons to the sclera so as to move the eye up or down and to the right or left. The procedure of the oblique muscles is more complicated. The superior oblique, which also arises in the posterior part of the socket, passes in the front of the eye through a loop or kind of pulley lying on the upper nasal side of the socket and then turns downwards to attach itself to the sclera. The inferior oblique muscle has its origin on the front lower nasal side of the eye-socket, and passes to the posterior surface of the eye-ball, being attached to the sclera on the temporal side. The superior oblique turns the eye downwards and outwards, and the inferior oblique turns it upwards and outwards.

The motor muscles of the two eyes act together so that both eyes turn always in the same sense, to the right or to the left, up or down. It is impossible to turn one eye up and the other down at same time, so as to look up to the sky with one eye and down at the earth with the other.

146. Optical Constants of the Eye.—The *optical axis* of the eye may be defined as the normal to the anterior surface of the cornea which goes through the center of the pupil. This line passes approximately through the centers of curvature of the refracting surfaces. The *schematic eye* (see § 130) is a centered system of spherical refracting surfaces symmetric with respect to the optical axis. The point where the optical axis meets the anterior surface of the cornea is called the *cornea vertex* or *anterior pole* of the eye and is designated

by **A**; and the point where the optical axis meets the retina is called the *posterior pole* of the eye and is designated by **B**. In GULLSTRAND'S schematic eye the distance from **A** to **B**. is equal to 24 mm., therefore somewhat less than an inch.

The motor muscles of the eye (§ 145), acting in pairs, turn the eye-ball around axes of rotation which all pass through a fixed point or pivot called the *center of rotation* of the eye and designated by **Z**. This point may be considered as lying also on the optical axis in the medium of the vitreous humor about 13 or 14 mm. from the vertex of the cornea or about 10.5 mm behind the pupil. All the excursions of the eye are performed around this point.

The object-point which is sharply imaged on the retina at the *fovea centralis* (§ 145) is called the *point of fixation*, and the straight line which joins the point of fixation with the centre of rotation is called the *line of fixation*. This line indicates the direction in which the eye is looking. The *field of fixation* is measured by the greatest angular distance through which the line of fixation can be turned; which amounts to about a right angle both vertically and horizontally.

In GULLSTRAND'S schematic eye, as was shown in § 130, the primary focal point **F** lies in front of the eye at a distance of 15.707 mm. from the anterior vertex of the cornea, while the secondary focal point **F'** lies on the other side of the cornea at a distance of 24.387 mm. The principal points (**H**, **H'**) lie in the aqueous humor slightly beyond the cornea system at distances $\mathbf{AH} = +1.348$ mm., $\mathbf{AH'} = +1.602$ mm. Thus the focal lengths are: $f = +17.055$ mm. $f' = -22.785$ mm.; the ratio between them being equal to 1.336, which is therefore the value of the index of refraction (n') of the vitreous humor. Accordingly, the refracting power of GULLSTRAND'S schematic eye is $F = 58.64$ dptr. The nodal points (**N**, **N'**) lie close to the posterior vertex of the crystalline lens, on opposite sides of it, at the following distances from the vertex of the cornea: $\mathbf{AN} = +7.078$ mm., $\mathbf{AN'} = +7.332$ mm. The straight line which joins the point

of fixation with the anterior nodal point of the eye is called the *visual axis*. It is parallel to the straight line which joins the posterior nodal point with the *fovea centralis*. Since the nodal points are so close together, for many problems connected with the eye they may be regarded as coincident; so that then the visual axis may be defined as the line drawn from the point of fixation to the *fovea centralis*. The visual axis meets the cornea a little to the nasal side of the anterior vertex and slightly above it, forming with the optical axis an angle between 3° and 5°.

The above values are all given for the passive, unaccommodated eye. By the act of accommodation the positions of the focal points, principal points and nodal points are all displaced, and accordingly the focal lengths and the refracting power of the eye can be varied within certain limits depending on the power of accommodation, as will be explained in the following section.

147. Accommodation of the Eye.—When the eye is at rest, as when one gazes pensively into space, it is adapted for far

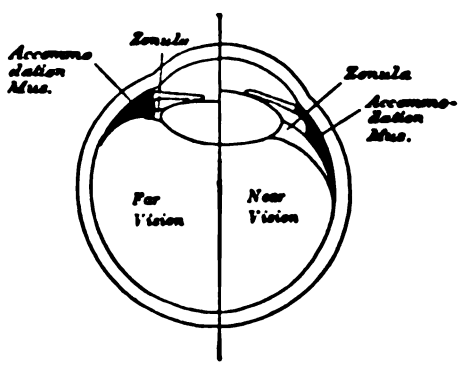


FIG. 193.—Accommodation of the human eye; indicating how the crystalline lens is changed from far vision to near vision.

vision, so that in order to see distinctly objects which are close at hand, an effort has to be made which will be greater in

proportion as the object fixed is nearer to the eye. This process whereby the normal eye is enabled to focus on the retina in succession sharp images of objects at different distances is called *accommodation*, and it is this marvelous adaptability of the human eye, together with its mobility, which perhaps more than any other quality entitles it to superiority over the most perfectly constructed artificial optical instruments. The power of accommodation is achieved by changes in the form of the crystalline lens, consisting chiefly in a change in the convexity of the anterior surface, produced through the mechanism of the ciliary muscle. According to the generally accepted theory, so long as the eye is passive, the elastic substance of the lens is held flattened in front by the suspensory ligament; but in the act of accommodation the ciliary muscle contracts, and this is accompanied by a relaxation of the ligament of the lens, which is thereby permitted to bulge forward by virtue of its own elasticity (Fig. 193).

148. Far Point and Near Point of the Eye.—The *far point* of the eye (*punctum remotum*) is that point (**R**) on the axis which is sharply focused at the posterior pole of the eye when the crystalline lens has its least refracting power; it is the point which is seen distinctly when the accommodation is entirely relaxed. On the other hand, the *near point* (or *punctum proximum*) is that point (**P**) on the axis which is seen distinctly when the crystalline lens has its greatest refracting power, that is, when the accommodation is exerted to the utmost. The region of distinct vision within which an object must lie in order that its image can be sharply focused on the retina of the naked eye is comprised between two concentric spherical surfaces, the far point sphere and the near point sphere, described around the center of rotation of the eye (**Z**) with radii equal to **ZR** and **ZP**, respectively. If the far point lies at infinity, as is the case in the normal eye, the far point sphere is identical with the infinitely distant plane of space (*cf.* § 83), as represented in Fig. 194;

whereas the near point sphere will be real and at a finite distance in front of the eye. In such a case the eye can be directed towards any point in the field of fixation (§ 146) lying on or beyond the near point sphere and accommodate

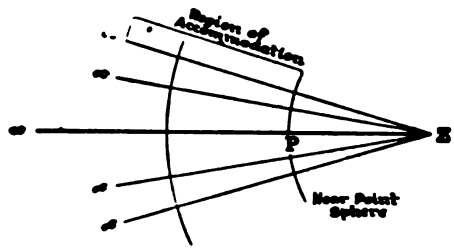


FIG. 194.—Region of accommodation of emmetropic eye.

itself to see this point distinctly. In a near-sighted eye both far point and near point are real points lying at finite distances in front of the eye; but the far point of a far-sighted eye is a "virtual" point lying at a finite distance behind the eye, and hence an unaided far-sighted eye cannot see distinctly a real object without exerting its accommodation to a greater or less degree.

149. Decrease of the Power of Accommodation with Increasing Age.—The faculty of accommodation is greatest in youth and diminishes rapidly with advancing years. The near point of the eye gradually recedes farther and farther away, which is commonly supposed to be due to a progressive diminution of the elasticity of the crystalline lens. Thus, at the ages of 10, 20 and 40 years the *punctum proximum* of a normal eye, according to DONDERS, is in front of the eye at distances from the primary principal point equal to 7.1, 10 and 22.2 cm., respectively. When the near point has retreated to a distance of 22 cm., so that it is no longer possible to read or write or do "near work" conveniently without the aid of spectacles, the condition of *presbyopia* or old-age vision has begun to set in. Meantime, while the

power of accommodation of the eye thus continually diminishes as the near point recedes farther and farther away, the position of the far point remains practically fixed until well after middle life; but between the ages of 55 and 60 years it too begins to separate farther from the eye, and thereafter both the near point and the far point travel outwards along the axis of the eye, the former, however, constantly gaining on the latter; until at last in extreme old age the near point actually overtakes the far point, and from that time until death they remain together, the power of accommodation having been entirely lost. Both points are displaced along the axis always in the same direction, that is, opposite to that of the incident light. For example, the far point of a normal eye is infinitely distant up to about 55 years of age, whereas ten years later, according to DONDERS, this point will be about 133 cm. behind the eye, having moved out through infinity, so to speak, and approached the eye from behind. At the same age, namely, 65 years, the near point will also be behind the eye at a distance of 400 cm. At 75 years of age the two points will be together at a distance of 57.1 cm. behind the eye. Various theories have been advanced to account for the senile recession of the far point of the eye. It is probably due to a combination of causes, partly to a change in the form of the lens produced by the increased resistance of the enveloping coat of the eye-ball and the decreased pressure of the surrounding tissue, and partly also to senile changes in the lens-substance itself whereby the "total index" of the lens is lowered in value.

150. Change of Refracting Power in Accommodation.— It was remarked above (§ 146) that the positions of the cardinal points of the optical system of the eye are all altered in the act of accommodation. Thus, for example, in GULLSTRAND'S schematic eye, which is calculated for an adolescent youth, the near point is at a distance $\mathbf{AP} = -10.23$ cm. *from* the vertex of the cornea; and for this state of maxi-

mum accommodation the positions of the focal points and principal points are found to be as follows:

$$\mathbf{AF} = -12.397 \text{ mm.}, \quad \mathbf{AF}' = +21.016 \text{ mm.},$$
$$\mathbf{AH} = +\ 1.772 \text{ mm.}, \quad \mathbf{AH}' = +\ 2.086 \text{ mm.};$$

and, accordingly, the focal lengths and the refracting power are:

$$f = +14.169 \text{ mm.}, \quad f' = -18.930 \text{ mm.}, \quad F = +70.57 \text{ dptr.}$$

It will be observed that, whereas the focal points have undergone considerable displacements from their positions in the passive eye, the corresponding displacements of the principal points are less than half a millimeter; and since in most physiological measurements half a millimeter is within the limit of error, we can usually afford to neglect altogether the accommodative displacement of the principal points of the eye, that is, we may regard the positions of the principal points **H**, **H′** as practically fixed and independent of the state of accommodation. This is one reason, among others, why the principal points of the eye have superseded the other cardinal points as points of reference. Their proximity to the cornea is another advantage, inasmuch as measurements referred to them are easily related to an external, visible and tangible point of the eye. In the so-called "reduced eye," which consists of a single spherical refracting surface separating the outside air from the vitreous humor and so placed that its vertex lies at the primary principal point of the schematic eye, the two principal points are, in fact, coincident with each other on the surface of this simplified cornea.

151. Amplitude of Accommodation.—*The far point distance* (***a***) and the *near point distance* (***b***) are the distances of the far point and near point, respectively, measured from the primary principal point of the eye; thus, $a = \mathbf{HR}$, $b = \mathbf{HP}$; it being tacitly assumed here that the position of the point **H** remains sensibly stationary during accommodation, as was explained above. Each of these distances is to be reckoned negative or positive according as the point in ques-

tion lies in front of the eye or behind it, respectively. The reciprocals of these magnitudes, namely, $A=1/a$, $B=1/b$, are termed the *static refraction* (A), or the refraction of the eye when the accommodation is completely relaxed, and the *dynamic refraction* (B), or the refraction of the eye when the accommodation is exerted to the highest degree. If the distances a and b are given in meters, the reciprocal magnitudes will be expressed in dioptries, as is generally the case.

The *range of accommodation* is defined to be the distance of the near point from the far point, that is, $\mathbf{RP}=b-a$; whereas the *amplitude of accommodation* is the value obtained by subtracting algebraically the magnitude of the dynamic refraction from that of the static refraction, thus:

$$\text{Amplitude of Accommodation}=A-B.$$

Imagine a thin convex lens placed in the primary principal plane of the eye with its axis in the same line as the optical axis of the eye, and of such strength that it produces at the far point of the eye an image of the near point; according to the above definition, the amplitude of accommodation of the eye is equal to the refracting power of this lens. For example, in the normal eye at 30 years of age, $a=\infty$, $b=-14.3$ cm., so that the amplitude of accommodation in this case amounts to 7 dptr.; whereas at 60 years of age $a=+200$ cm., $b=-200$ cm., and hence the amplitude of accommodation will have been reduced to 1 dptr.

The distance from the secondary principal point ($\mathbf{H}'$) to the posterior pole ($\mathbf{B}$) where the optical axis meets the retina may be regarded as a measure of the length of the eye-axis, especially since the position of $\mathbf{H}'$ is sensibly independent of the state of accommodation, as has been explained, (§ 150). If this distance is denoted by a', that is, if we put $a'=\mathbf{H}'\mathbf{B}$, and if also we put $A'=n'/a'$, where n' denotes the index of refraction of the vitreous humor, then we may write:

$$A'=A+F,$$

where F denotes here the refracting power of the passive, unaccommodated eye. Similarly, if the symbol F_a is em-

ployed to denote the refracting power of the eye in its state of maximum accommodation, we shall have:

$$A' = B + F_a.$$

Consequently, we may also say that the power of accommodation $(A - B)$ is equal to the difference $(F_a - F)$ between the greatest and least values of the refracting power of the eye.

152. Various Expressions for the Refraction of the Eye.—The *refraction of the eye* in a given state of accommodation is measured by the reciprocal of the distance from the eye of the axial object-point M for which the eye is accommodated. Thus, if $u = \mathbf{H}M$, $x = \mathbf{F}M$ denote the distances of M from the primary principal point and the primary focal point, respectively, the magnitudes $U = 1/u$ and $X = 1/x$, usually expressed in dioptries, are the measures of the principal point refraction and the focal point refraction. The relation between U and X may be given in terms of the refracting power of the eye (F) when it is accommodated for the object-point M, as follows:

$$U = \frac{F.X}{F - X}, \quad X = \frac{F.U}{F + U}.$$

If an arbitrary point O on the axis of the eye is selected as the point of reference, and if we put $OM = z$, the refraction of the eye, referred to the point O, will be measured by $Z = 1/z$. If the distances of the points $\mathbf{H}$ and $\mathbf{F}$ from O are denoted by b and g, that is, if $b = O\mathbf{H}$, $g = O\mathbf{F}$, then since $z = u + b = x + g$, we can obtain also the following useful relations between U, X and Z in terms of b and g:

$$U = \frac{Z}{1 - b.Z} = \frac{X}{1 - (b - g)X};$$

$$X = \frac{Z}{1 - g.Z} = \frac{U}{1 + (b - g)U};$$

$$Z = \frac{U}{1 + b.U} = \frac{X}{1 + g.X}.$$

153. Emmetropia and Ametropia.—When the static refraction of the eye is equal to zero $(A = 0)$, that is, when

the far point (**R**) is infinitely distant, the eye is said to be *emmetropic*. If in the equation $A'=A+F$, we put $A=0$, we obtain $A'=F$, which therefore may be said to be the condition of emmetropia. Here F denotes the refracting power of the eye when accommodation is entirely relaxed. In emmetropia, therefore, the second focal point (**F'**) of the passive eye lies on the retina at the posterior pole (**B**);

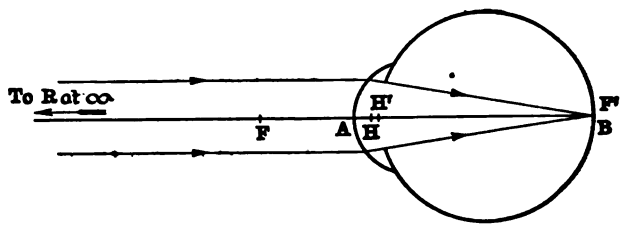


FIG. 195.—Diagram of emmetropic eye.

so that in a passive emmetropic eye incident parallel rays are converged to a focus on the retina, as represented in Fig. 195, and the length of the eye-axis is $a'=-f'$. The normal position of the far point is to be regarded as at infinity; and in this sense an emmetropic eye is a normal eye, although, strictly speaking, an emmetropic eye may at the same time be abnormal in various ways.

On the other hand, if the static refraction of the eye is different from zero ($A\neq 0$), that is, when the far point (R) is not infinitely distant, the eye is said to be *ametropic* Thus, the condition of *ametropia* may be said to be characterized by the fact that the refracting power (F) of the unaccommodated eye is not equal to A', which is equivalent to saying that the length of the eye-axis (a') is numerically different from the value of the second focal length (f'). In other words, the second focal point (**F'**) of an ametropic eye in a state of repose does not fall on the retina.

Two general divisions of ametropia are distinguished depending, on whether the far point (**R**) lies on one side or the

other of the primary principal point (H). Thus, if $A<0$, that is, if the far point lies at a finite distance in front of the eye, the ametropia in this case is called *myopia* (Fig. 196). In a myopic eye in a state of repose the second focal point

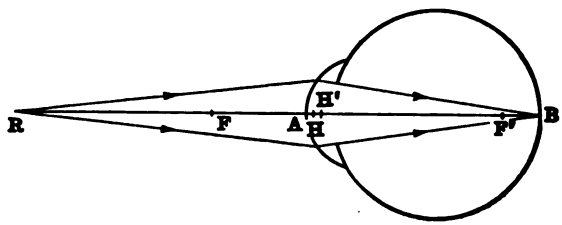


FIG. 196.—Ametropic eye: myopia.

(F') lies in front of the retina (in the vitreous humor), so that parallel incident rays will be brought to a focus before reaching the retina. On the other hand, if $A>0$, the far point will lie at a finite distance beyond (or behind) the

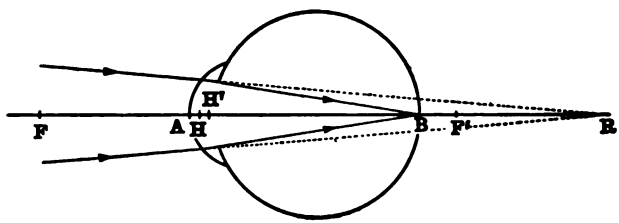


FIG. 197.—Ametropic eye: hypermetropia.

eye, and this form of ametropia is known as *hypermetropia* (Fig. 197). In a hypermetropic eye in a state of repose the second focal point (F') falls beyond the retina, so that incident parallel rays arrive at the retina before coming to a focus. A myopic eye cannot focus for a distant object without the aid of a glass, and it lacks therefore an important part of the capacity of an emmetropic eye. On the other hand, a hypermetropic eye must make an effort of accommodation each time in order to focus on the retina the image

of a real object; which frequently produces various troubles, sometimes very annoying. Accordingly, both conditions included under the general name of ametropia are disadvantageous for practical vision.

Theoretically, ametropia may be considered as due to some abnormality in the values of one or of both of the magnitudes denoted by A' and F' on which the value of the static refraction (A) depends; so that the following cases are possible:

(1) The length of the eye-ball (a') may be too great (axial myopia, $a' > -f'$) or too small (axial hypermetropia, $a' < -f'$), whereas the refracting power (F) is normal. This, by far the most common, type is known as *axial ametropia*.

(2) On the other hand, while the length of the eye-ball may be normal, the magnitude of the refracting power (F) may be abnormally great or small. In general, this form of ametropia, which is comparatively rare, is due to abnormal curvatures of the refracting surfaces (*curvature ametropia*). Or the indices of refraction of the eye-media may have abnormal values (*indicial ametropia*). Here also may be mentioned the condition known as *aphakia* produced by the extraction of the crystalline lens in the operation for cataract.

(3) Finally, it may happen that the refracting power and the length of the eye-ball are both abnormal. In fact, these two anomalies might exist together in exactly the degree necessary to counteract each other, so that, in spite of its abnormalities, the eye in such a case would be emmetropic.

In the case of axial ametropia, the relation between the static refraction (A) and the length (l) of the eye-ball is given by the following formula:

$$l = \mathbf{AB} = \mathbf{AH} + \frac{n'}{A+F};$$

and if the values for GULLSTRAND'S schematic eye (§ 146) are substituted in this formula, it may be written as follows:

$$l = 1.602 + \frac{1336}{A+58.64} \text{ millimeters.}$$

According to this formula, the length of the eye varies from about 21.07 mm. in extreme axial hypermetropia ($A = +10$ dptr.) to about 36.18 mm. in case of the highest degree of axial myopia ($A = -20$ dptr.). The length of an axially emmetropic eye ($A = 0$) is 24.38 mm. The length of GULLSTRAND'S schematic eye is 24.01 mm., and hence this eye has 1 dptr. of hypermetropia ($A = +1$ dptr.). The accompanying diagram (Fig. 198) exhibits graphically the relation between the magnitudes denoted by l and A. The heights of the ordinates indicate the axial length of the eye-ball in millimeters for values of the static refraction of the eye comprised between -20 and $+10$ dioptries.

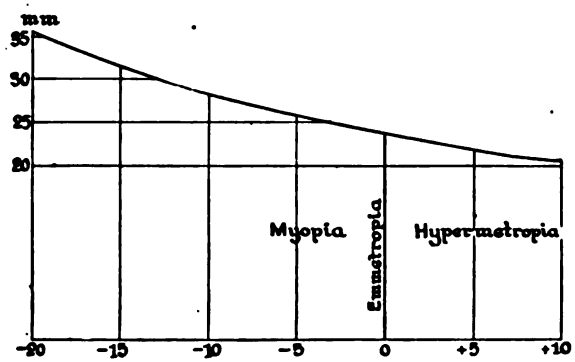


FIG. 198.—Curve showing connection between the length of the eye-axis and the static refraction.

154. Correction Eye-Glasses.—When a spherical spectacle lens is placed in front of the passive, unaccommodated eye, with the axis of the lens in the same straight line as the optical axis of the eye, there will be a certain axial point M whose image in the lens will fall at the far point (R) of the eye; and hence the eye looking through the lens will see distinctly the image of an object placed at M. If the positions of the

principal points of the lens are designated by H_1 and H_1', and if we put

$$u_1 = 1/U_1 = H_1M, \quad u_1' = 1/U_1' = H_1'R,$$

then

$$U_1' = U + F_1,$$

where F_1 denotes the refracting power of the lens. Let the distance of the primary principal point (H) of the eye from the secondary principal point (H_1') of the lens be denoted by c, that is, $c = H_1'H$; then since $a = u_1' - c$, where a denotes the far point distance of the eye, the following expression for the static refraction ($A = 1/a$) may be derived immediately:

$$A = \frac{U_1 + F_1}{1 - c(U_1 + F_1)}.$$

In case the axial object-point M is infinitely far away, the lens is called a *correction-glass*, because it enables the passive ametropic eye to see distinctly a very distant object on the axis of the lens, so that to this extent the lens interposed in front of the eye endows it with the characteristic faculty of an unaccommodated, naked, emmetropic eye. The condition that M shall be infinitely distant is $U_1 = 0$; and hence the relation between the static refraction of the eye and the refracting power of a correction-glass is given as follows:

$$A = \frac{F_1}{1 - c.F_1} \qquad F_1 = \frac{A}{1 + c.A}.$$

If the distance c between the correction-glass and the eye is neglected entirely, then $F_1 = A$, that is, the power of the correction-glass is approximately equal to the static refraction of the eye. The distance c, which must be expressed in meters in case the magnitudes denoted by F_1 and A are given in dioptries, is always a comparatively small magnitude, which in actual spectacle glasses is comprised between 0.008 and 0.016 m.; so that if, without neglecting c entirely,

we neglect only the second and higher powers thereof, the formulæ above may be written in the following convenient approximate forms:

$$A = F_1(1+c.F_1), \qquad F_1 = A(1-c.A);$$

which for nearly all practical purposes will be found to be sufficiently accurate.

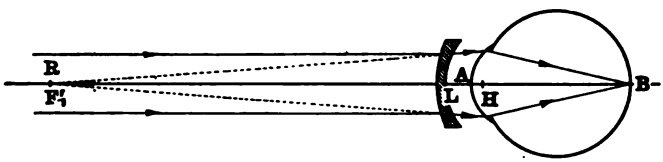


FIG. 199.—Correction of myopia with concave spectacle-glass.

The condition that a spectacle-lens shall be a correction-glass may be expressed simply by saying that *the second focal point* (F_1') *of the glass must coincide with the far point* (**R**) *of the eye.* Thus, in case of a myopic eye the correction-glass

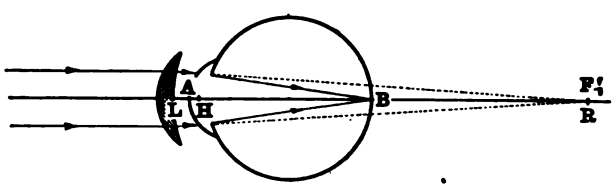


FIG. 200.—Correction of hypermetropia with convex spectacle-glass.

will be concave (Fig. 199) and in case of a hypermetropic eye it will be convex (Fig. 200).

Instead of describing the power of a spectacle glass by means of its refracting power, it is really more convenient and logical to express it in terms of its *vertex refraction* (V), as defined in § 128. If the vertex of the lens which lies next the eye is designated by L, and if the distance of the eye from the glass is denoted by k, that is, if we put $k = \mathrm{LH}$, then, since the points designated by F_1' and **R** must be coincident,

$v=a+k$, where a denotes the "back focus" of the lens, that is, $v=1/V=LF_1'=LR$; and hence:

$$A=\frac{V}{1-k.V}, \qquad V=\frac{A}{1+k.A},$$

or approximately:

$$A=V(1+k.V), \qquad V=A(1-k.A).$$

It may be seen from the above formulæ how the power of a correction-glass depends essentially on the location of the glass in front of the eye. The distance k, being referred to a tangible, external point of the glass, is more easily measured than the interval denoted by c.

155. Visual Angle.—The apparent size of an object, as was explained in § 10, is measured by the visual angle ω which it subtends at the eye; thus, if the vertex of this angle is designated by O and if $y=MQ$ denotes a diameter of the object at right angles to the line of vision, the apparent size of the object in the direction of this dimension is $\omega=\angle MOQ$. Accordingly, if the distance of the object from the eye is denoted by z, that is, if $z=OM$, then $\tan\omega=y/z$. As the immobile eye looking in a fixed direction can see distinctly only that comparatively small portion of the object whose image falls on the sensitive part of the retina in the immediate vicinity of the *fovea centralis* (§ 145), the rays concerned in the production of the retinal image in this so-called case of "indirect vision" may be regarded as paraxial rays. Accordingly, the value of the angle ω in radians may be substituted here for the $\tan\omega$, so that we may write:

$$\omega=y/z=y.Z,$$

where $Z=1/z$. On the assumption that y is reckoned as positive, a negative value of the angle ω indicates that the object is real and therefore in front of the point O where the eye is supposed to be.

The exact meaning to be attached to the visual angle ω will depend, of course, on the precise location with respect to the eye of the vertex of this angle. To be sure, so long as the object is quite remote from the eye, as is often the case,

it will not generally be necessary to define particularly the position of the vertex O of the visual angle. For example, to take a somewhat extreme instance, the apparent size of the moon will not be sensibly altered by removing the vertex of the visual angle as much as a mile or more away from the eye. And, in general, provided the object is not less than, say, 10 meters away, it will be sufficient to know that the vertex of the visual angle is in the eye without specifying its position more exactly. On the other hand, especially when the eye has to exert its power of accommodation in order to focus the object, it is sometimes a matter of much importance to define the visual angle with the utmost precision. In such a case several meanings of this term are to be specially distinguished. For example, when the vertex of the visual angle is at the primary principal point of the eye, it is called the *principal point angle* ($\omega_H = \angle MHQ$), so that we may write:

$$\omega_H = y/u = y.U,$$

where $u = 1/U = HM$ denotes the distance of the object from the primary principal point. Similarly, the so-called *focal point angle* ($\omega_F = \angle MFQ$) is the angle subtended by the object at the primary focal point of the eye; and hence:

$$\omega_F = y/x = y.X,$$

where $x = 1/X = FM$ denotes the distance of the object from the primary focal point of the eye.

According to the definitions of these angles and the relations between the magnitudes denoted by X, U and Z, as given in § 152, we may write therefore:

$$\begin{aligned} \omega : \omega_H : \omega_F = Z : U : X \\ = 1 : (1+b.U) : (1+g.X) \\ = (1-b.Z) : 1 : (1-X/F) \\ = (1-g.Z) : (1+U/F) : 1; \end{aligned}$$

where F denotes here the refracting power of the eye when it is accommodated for the point M.

The apparent size of an object may be measured also at other points of the eye, for example, at the center of the

entrance-pupil, at the anterior nodal point, at the center of rotation, etc. The center of rotation or eye-pivot is the point of reference in the estimate of the apparent size of an object in the case of ordinary so-called "direct vision" with the mobile eye, when the gaze is directed in quick succession to the different parts of an extended object. Especially, in viewing an image through an optical instrument, it is nearly always desirable, if practicable, to adjust the eye in such a position that the center of rotation coincides with the center of the exit-pupil of the instrument, so as to command as large an extent of the field of view of the image-space as possible. Anyone who has ever tried to look through a key-hole in a door will realize how the field of view would have been widened if the eye could have been placed in the hole itself.

156. Size of Retinal Image.—If the eye is accommodated to see an object y situated at a distance u $(=1/U)$ from its primary principal point, the size of the image (y') formed on the retina is given by the relation:

$$y.U = y'.A',$$

where $A' = n'/a'$ denotes the reciprocal of the reduced length of the eye-axis measured from the secondary principal point of the eye. Since $y.U = \omega_H$ (§ 155), the above equation may be put in the following form:

$$\frac{y'}{\omega_H} = \frac{a'}{n'}.$$

Since the positions of the principal points remain sensibly stationary in the act of accommodation (§ 150), the reduced length of the eye-axis (a'/n') may be considered as constant in the same individual. And hence the peculiar significance of the principal point angle consists in the fact that, according to this formula, this angle (ω_H) may be taken as a measure of the size of the retinal image (y') which is independent of the state of accommodation of the eye. Thus, for a given individual, all objects which have the same ap-

parent size as measured at the principal point of the eye will produce retinal images of equal size.

On the other hand, since $y'.F = y.X = \omega_F$ (§ 155), it appears that, for a given value of the refracting power (F), the size of the image on the retina of the eye is proportional to the focal point angle. And since the variations of the refracting power are, generally speaking, independent of axial ametropia (§ 153), the focal point angle will be particularly useful in comparing the apparent size of an object as seen by different individuals under the same external conditions.

157. Apparent Size of an Object seen Through an Optical Instrument.—Let the principal points of the optical instrument be designated by H, H′ (Fig. 201); and for the

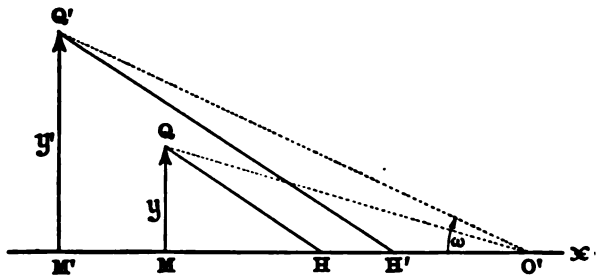


Fig. 201.—Apparent size of object seen through an optical instrument.

sake of simplicity, let us assume that the instrument is surrounded by air so that the straight lines HQ, H′Q′ joining the principal points with corresponding points of object and image will be parallel; and let $y = MQ$, $y' = M'Q'$ denote the linear magnitudes of object and image, respectively. Let the distance of the image from the eye be denoted by $z = O'M'$, where O′ designates the position of the eye on the axis. Then the apparent size of the image will be

$$\omega = y'.Z,$$

where $\omega = \angle M'O'Q'$ (expressed here in radians) and $Z = 1/z$. The angle ω may be increased by reducing the distance be-

tween the image and the eye, that is, by increasing Z; but this distance cannot be diminished below the near point distance of the eye, because then distinct vision would not be possible for the naked eye.

If the distances of object and image from the principal points are denoted by u and u', that is, if $u = \mathrm{HM}$, $u' = \mathrm{H'M'}$, then

$$y'.U' = y.U,$$

where $U = 1/u$, $U' = 1/u'$; and hence

$$\omega = y.Z.\frac{U}{U'}.$$

In general (except when the rays undergo an odd number of reflections), the sign of Z as here defined will be negative, and therefore the sign of ω will depend on the sign of the ratio $U : U'$. Accordingly, if object and image lie on the same side of their corresponding principal points, the sign of ω will be negative, that is, the image will be erect.

Let the distance of the eye from the instrument be denoted by $c = \mathrm{H'O'}$; then since $u' = c + z$, we may write:

$$U' = \frac{Z}{1 + c.Z}.$$

Accordingly, if the refracting power of the instrument is denoted by F, so that $U = U' - F$, we may write also:

$$U = -\frac{F - Z(1 - c.F)}{1 + c.Z}.$$

Introducing these expressions, we obtain therefore the following formula for the apparent size of the image:

$$\omega = -y\{F - Z(1 - c.F)\}.$$

Thus, we see that the apparent size of the image may be varied in one of two ways, either by changing the position of the eye (that is, by varying c) or else by displacing the object so that Z is varied. There are two cases of special practical importance, namely: (1) When the eye is adjusted so that $1 - c.F = 0$, and (2) When the object is focused so that $Z = 0$. In both cases the second term inside the large brackets vanishes, and hence $\omega = -y.F$. The condition

$c = -1/F$ means that the eye is placed at the second focal point (F′) of the instrument (which might easily be practicable if the optical system were convergent); so that under such circumstances the apparent size of the image would be the same for all positions of the object, because evidently the highest point (Q′) of the image will always lie on the straight line which crosses the axis at the second focal point at the constant angle $\theta = -y.F$. On the other hand, the condition $Z = 0$ means simply that the object lies in the first focal plane of the instrument. Now this is the natural adjustment for a normal, unaccommodated, emmetropic eye, because then the rays flow into the eye in cylindrical bundles. This is the reason why the image produced by the object-glass of a telescope or microscope is usually focused in the primary focal plane of the eye-piece or ocular. Accordingly, when $Z = 0$, the apparent size of the image will be independent of the position of the eye.

An experienced observer who wishes to obtain the best results with an optical instrument will ordinarily adjust it to his eye in such a way that the image can be seen distinctly without his having to make an effort of accommodation. This will be the case if the image is formed at the far point (R) of the eye (§ 148). If, therefore, the static refraction of the eye is denoted by A (§ 151), then (assuming that the point O′ in Fig. 201 is coincident with the anterior principal point of the eye) we may put $Z = A$; and hence the apparent size of an object as seen in an optical instrument by an eye with relaxed accommodation is given by the expression:

$$\omega_H = -y\{F - A(1 - c.F)\}.$$

Thus, it is evident how the apparent size of the image depends not only on the refracting power of the instrument, but essentially also on the adjustment and idiosyncrasies of the eye of the individual who looks through it.

It may be remarked that these formulæ have been derived on the tacit assumption that the eye is at rest, and consequently only a small portion of the external field is sharply

in focus at the sensitive part of the retina. Otherwise, we should have had to write $\tan\omega$ instead of ω; nor should we have been justified in assuming that the effective rays were paraxial. If the eye turns in its socket to inspect the image, the apparent size of the image will depend essentially on the angular movement of the eye, and in this case the visual angle must be measured at the center of rotation of the eye. These are considerations that are too often overlooked in discussions of this kind.

158. Magnifying Power of an Optical Instrument Used in Conjunction with the Eye.—An object may be so remote that its details are indistinguishable, or, on the other hand, it may be so close to the eye that not even by the greatest effort of accommodation can a sharp image of it be focused on the retina. Under such circumstances one has recourse to the aid of a suitable optical instrument whereby the object is magnified to such an extent that the parts of it which were obscure or entirely invisible to the naked eye will be revealed to view. The *magnifying power* is usually expressed by an abstract number M, which in the case of an optical instrument on the order of a microscope is defined to be *the ratio of the apparent size of the image as seen in the instrument to the apparent size of the object as it would appear at the so-called "distance of distinct vision."* This latter term is a somewhat unfortunate form of expression for several reasons, not only because the distance at which an object is ordinarily placed in order to be seen distinctly is different for different persons, but because the same person, according to the extent of his power of accommodation, usually possesses the ability of seeing distinctly objects at widely different distances. The expression appears to have arisen from a confusion of ideas, and its origin may probably be traced to the fact that even nowadays many people have difficulty in conceiving how the eye can be "focused for infinity," although, indeed, as has been explained, that is to be regarded as the natural state of the normal eye in re-

pose. However, the phrase has become too deeply rooted in optical literature ever to be eradicated, and no harm will be done by continuing to use it, provided it is not taken literally, but is considered merely as the designation of a more or less arbitrary conventional projection-distance. Accordingly, if the so-called "distance of distinct vision" is denoted by l, the apparent size of the object (y) as seen at this distance from the eye will be $-y/l$, and hence if the apparent size of the image in the instrument is denoted by ω, the magnifying power, as above defined, will be:

$$M = -\frac{\omega}{y} . l .$$

The actual value of this conventional distance l is usually taken as 10 inches or 25 centimeters, which is large enough for the convenient accommodation of most human beings who are not already past the prime of life and yet not so large that the size of the image on the retina differs much from its greatest dimensions. If distances are all measured in meters, the conventional value of the magnifying power will be given, therefore, by the formula:

$$M = -\frac{\omega}{4y} .$$

The explanation of the minus sign in front of the fraction is to be found in the mode of reckoning the visual angle ω, which, as we have pointed out (§ 157), is negative in case the image of the object y is erect, as, for example, with an ordinary convex lens used as a magnifying glass. Thus, according to the above formula, *a positive value of the magnifying power means magnification without inversion.* Ordinarily, what is meant by the magnifying power of an optical instrument is the value of this abstract number M; which gives the ratio of the sizes of the retinal images when an emmetropic eye views one and the same object, first, in the instrument without effort of accommodation, and then without the instrument with an accommodation of four dioptries.

If the expression for the visual angle ω which was ob-

tained in § 157 is introduced here, we shall derive, therefore, the following formula for the magnifying power (M) in terms of the refracting power (F) of the instrument, the distance (c) of the eye from the instrument, and the distance ($z = 1/Z$) of the image (y') from the eye:

$$M = l\{F - Z(1 - c.F)\}.$$

This expression is really a measure of the *individual magnifying power*, since it involves not merely the instrument itself but the characteristic peculiarities of the eye of the observer. In order to obtain a measure of the *absolute magnifying power* of the instrument, the second term inside the large brackets must be made to vanish. Thus, if the object is placed in the primary focal plane, so that the image is infinitely distant, then $Z = 0$, and now $M = l.F$ denotes the absolute magnifying power. If $l = 0.25$ meter, then $F = 4M$; and usually, therefore, when we say that the magnifying power of a lens or microscope is M, this means simply that its refracting power is equal to 4M dioptries.

If the image in the instrument is formed at the "distance of distinct vision" (l), then $Z = -1/l$. and

$$M = 1 + (l - c)F.$$

The distance (c) between the instrument and the eye is usually small in comparison with l, so that it is often entirely neglected. Assuming that $(l - c)$ is positive, we may say that in a convergent optical system ($F > 0$), the object will appear magnified ($M > 1$); whereas in a divergent optical system ($F < 0$), the object appears to be diminished in size ($M < 1$).

In order to avoid the use of an arbitrary projection-distance, (l), ABBE proposed to define the magnifying power as *the ratio of the apparent size* (ω) *of the image in the instrument to the actual size* (y) *of the object* (compare with ABBE's definition of focal length, § 122); so that if this ratio is denoted by P, then

$$P = -\frac{\omega}{y}.$$

This measure of the magnifying power is not an abstract number like M, but a quantity of the same physical dimensions as the refracting power of the instrument. The two definitions are connected by the simple relation

$$M = l.P;$$

so that if we put $l = 0.25$ m., the value of P will be obtained by multiplying M by the number four ($P = 4M$). Thus, for example, in the case of a convex lens of refracting power F used as a magnifying glass, if the object is placed in the first focal plane, we have $P = F$.

159. Magnifying Power of a Telescope.—In the case of a telescope, which is an instrument for magnifying the apparent size of a distant object, neither of the definitions of magnifying power given in the foregoing section is applicable. An infinitely distant object (like the moon, for example) can be seen distinctly by an emmetropic eye without any effort of accommodation, but its apparent size may be so minute that the distinguishing features cannot be made out by the naked eye. This same eye looking at the object through a telescope will see an infinitely distant image of it, but presented to the eye under a larger visual angle, so that it appears magnified. Essentially, a telescope may be regarded as a combination of two optical systems, one of which—the part pointed towards the object—is a convergent system, generally of relatively long focus and large aperture (so as to intercept a large quantity of light), called the object-glass; while the other, composed of the lenses next the eye, and called therefore the ocular or eye-piece, may be a convergent or divergent system depending on the type of telescope. The object-glass which is at one end of a large tube forms a real inverted image of the object in its second focal plane or not far from it; and this image is inspected through the ocular, which is usually fixed in a smaller tube inserted in the larger one so that the focus can be adjusted to suit different eyes and different circumstances. A simple schematic telescope may be regarded as composed

of two thin lenses, one of which, of focal length f_1 (refracting power F_1) acts as the object-glass while the other, of focal length f_2 (refracting power F_2) performs the part of the ocular. When the telescope is adjusted for an emmetropic, unaccommodated eye, *the second focal point* (F_1') *of the object-glass coincides with the first focal point* (F_2) *of the ocular;* and hence the focal length of the entire system is infinite ($f = \infty$ or $F = 0$), that is, the system is afocal or telescopic (§ 125). In this case the telescope is said to be in normal adjustment.

The first telescope appears to have been invented by one of two Dutch spectacle-makers named ZACHARIAS JANSEN and FRANZ LIPPERSHEY (*circa* 1608). GALILEO (1564–1642), having heard of this Dutch toy, was led to experiment

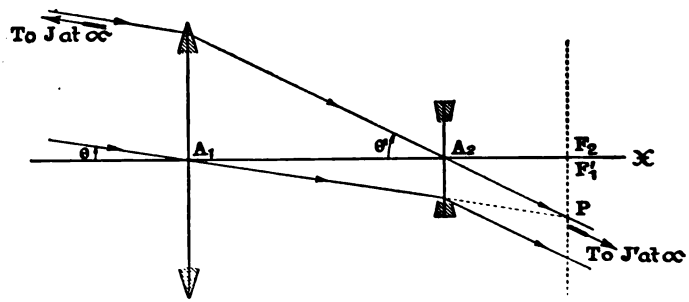


FIG. 202.—Diagram of simple Dutch or Galilean telescope.

with a combination of two lenses and he soon succeeded (1609) in making a telescope with which he made a number of renowned astronomical discoveries. The so-called *Dutch* or *Galilean telescope*, represented schematically in Fig. 202, consists of a large convex object-glass (A_1) combined with a small concave eye-piece (A_2), which intercepts the converging rays before they come to a focus and adapts them to suit the eye of the observer. The other type of telescope (Fig. 203) is composed of two convex lenses. It is called the *astronomical telescope* or *Kepler telescope*, because the idea

occurred first to JOHN KEPLER (1611); but the first instrument of this kind was made by the celebrated Jesuit father, CHRISTIAN SCHEINER (1615), who also conceived the idea of using a third lens to erect the image as is done in the so-called *terrestrial telescope*.

If the telescope is in normal adjustment, then from each point J of the infinitely distant object there will issue a bundle

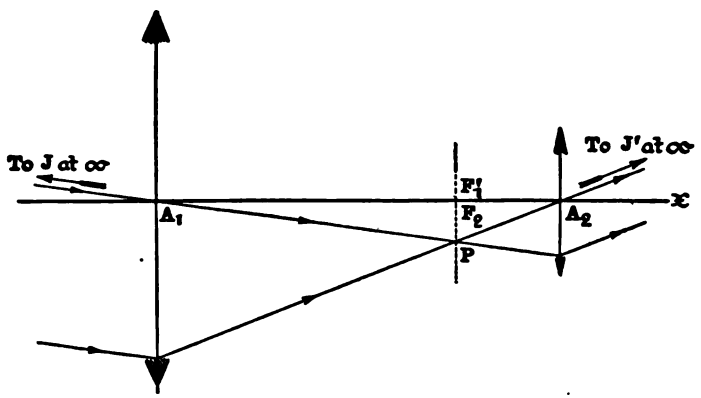


FIG. 203.—Diagram of simple astronomical telescope.

of parallel rays whose inclination to the axis of the telescope may be denoted by θ. Falling on the object-glass, these rays are converged to a focus at a point P lying in the common focal plane of object-glass and eye-piece; and consequently they will emerge from the eye-piece and enter the eye as a bundle of parallel rays proceeding from the infinitely distant image-point J′ in a direction which makes an angle θ' with the axis. The slope-angles θ and θ' have a constant relation to each other, as may easily be shown; for from the right triangles $F_1'A_1P$ and F_2A_2P (Figs. 202 and 203), where $A_1F_1' = A_1F_2 = f_1$, $\angle F_1'A_1P = \theta$, and $F_2A_2 = F_1'A_2 = f_2$, $\angle F_2A_2P = \theta'$, we obtain immediately:

$$\frac{\tan\theta'}{\tan\theta} = -\frac{f_1}{f_2} = \text{constant}.$$

Now the angles denoted here by θ and θ' are the measures of the apparent sizes of corresponding portions of the infinitely distant object and image, and the ratio of these angles (or of their tangents) is defined to be the *magnifying power of the telescope;* so that if this ratio is denoted by M, we shall have:

$$M = -\frac{f_1}{f_2}.$$

Accordingly, the magnifying power of a telescope focused on an infinitely distant object and adjusted for distinct vision for an unaccommodated, emmetropic (or corrected ametropic) eye is measured by the ratio of the focal lengths of the objective and ocular. In the astronomical telescope f_1 and f_2 are both positive, and consequently the ratio M is negative, which means that the image is inverted; whereas in the Dutch telescope f_1 is positive and f_2 is negative, and hence M is positive, that is, the final image is erect.

Another convenient expression for the magnifying power of a telescope, as defined above, may easily be obtained. All the effective rays which fall on the object-glass will after transmission through the instrument pass through a certain circular aperture called the *eye-ring* (or RAMSDEN circle), which is the image of the object-glass in the ocular. If the object-glass is brightly illuminated (for example, if the telescope is pointed towards the bright sky), this image appears as a luminous disk floating in the air not far from the ocular and can easily be perceived by placing the eye at a suitable distance. In the astronomical telescope the eye-ring is a real image which can be received on a screen, and in this instrument it usually acts as the exit-pupil (§ 134). In the case of the Dutch telescope the eye-ring is a virtual image on the other side of the ocular from the eye; and generally its effect is to limit the field of view in the image-space, that is, its office is that of the exit-port of the system (§§ 137, 138). Now if the telescope is in normal adjustment, then the distance of the ocular from the object-glass is equal to

the algebraic sum (f_1+f_2) of the focal lengths of the two components; and it may easily be shown that

$$M = -\frac{f_1}{f_2} = \frac{\text{diameter of object-glass}}{\text{diameter of eye-ring}}.$$

The advantage of this latter form of expression is to be found in the fact that even if the telescope is not in normal adjustment, it may still be considered in a certain sense as a measure of the magnifying power of the instrument. Suppose, for example, that the optical system is not telescopic, so that the interval between the second focal point (F_1') of the object-glass and the first focal point (F_2) of the ocular is not negligible, as frequently happens in focusing the eye-piece to suit the eye of the individual, especially if the object itself is not infinitely distant. Consider a ray which is directed originally from the extremity of the object towards a point O on the axis of the telescope and which emerges so as to enter the eye at the conjugate point O'. If the angles which the ray makes with the axis at O and O' are denoted by θ and θ', respectively, then the ratio $\tan\theta' : \tan\theta$ will be a measure of the magnifying power of the telescope for this adjustment and position of the eye. But according to the Smith-Helmholtz formula (§§ 86 and 118), since the telescope is surrounded by the same medium on both sides, we shall have here:

$$\tan\theta' : \tan\theta = y : y',$$

where y and y' denote the linear magnitudes of an object and its image in conjugate transversal planes at O and O' (the planes of the pupils). Now if the point O' is at the center of the eye-ring, the point O will lie at the center of the object-glass, and the ratio $y : y'$ will be equal to the ratio of the diameters of object-glass and eye-ring. Hence, *provided the eye is placed at the eye-ring*, the magnifying power of the telescope will be

$$M = \frac{\text{diameter of object-glass}}{\text{diameter of eye-ring}}.$$

In an astronomical telescope the best adjustment for com-

manding a wide extent of the field of view is to place the eye with its center of rotation at the center of the eye-ring, but in a Dutch telescope this is not practicable, because the eye-ring is not accessible.

In order to obtain a general formula for the magnifying power of a telescope, let us fix our attention on the inverted image of the object which is formed by the object-glass. If $u=1/U$ denotes the distance of the object from the object-glass and if q denotes the linear size of the image, the apparent size of the object as seen from the center of the object-glass will be

$$\tan\theta=q(U+F_1),$$

where F_1 denotes the refracting power of the object-glass. On the other hand, according to the formula deduced in § 157, the apparent size of the image seen in the telescope will be

$$\tan\theta'=-q\{F_2-Z(1-c.F_2)\},$$

where F_2 denotes the refracting power of the ocular, $z=1/Z$ denotes the distance of the image in the ocular from the eye, and c denotes the distance of the eye from the ocular itself (or from its second principal point). Accordingly, we obtain the following expression for the magnifying power of the telescope:

$$\mathrm{M}=\frac{\tan\theta'}{\tan\theta}=-\frac{F_2-Z(1-c.F_2)}{U+F_1},$$

which is applicable to all cases. If the object is infinitely distant, then $U=0$; and if the telescope is in normal adjustment, then the image is also infinitely distant, that is, $Z=0$, and $\mathrm{M}=-F_2/F_1$.

PROBLEMS

1. If the refracting power of a correction spectacle-glass is +10 dptr., and if the distance of the anterior principal point of the eye from the second principal point of the glass is 12 mm., find the static refraction of the eye.

Ans. +11.36 dptr.

2. Take the refracting power of the eye equal to 58.64 dptr., the distances of the principal points from the vertex of the cornea as 1.348 and 1.602 mm., and the index of refraction of the vitreous humor equal to 1.336. If the refracting power of a correction spectacle-glass, whose second principal point is 14 mm. from the anterior principal point of the eye, is +5.37 dptr., show that the total length of the eye-ball is 26.5 mm.

3. In GULLSTRAND'S schematic eye, with accommodation relaxed, the distance from the vertex of the cornea to the point where the optical axis meets the retina is 24 mm. The other data are the same as those given in No. 2 above. Find the position of the far point and determine the static refraction.

Ans. The far point is 99.34 cm. from the vertex of the cornea, and the static refraction is +1.008 dptr.

4. In GULLSTRAND'S schematic eye in its state of maximum accommodation the distances of the principal points from the vertex of the cornea are 1.7719 and 2.0857 mm., and the refracting power is 70.5747 dptr. The length of the eye-ball is 24 mm., as stated in No. 3. Find the position of the near point and determine the dynamic refraction of the eye.

Ans. The near point is 10.23 cm. from the vertex of the cornea; the dynamic refraction is −9.609 dptr. Accordingly, with the aid of the result obtained in No. 3, we obtain for the amplitude of accommodation 10.62 dptr.

5. Taking the refracting power of the eye as equal to 59 dptr., show that the size of the retinal image of an object

1 meter high at a distance of 10 meters from the eye will be 1.7 mm.

6. The apparent size of a distant air-ship is one minute of arc. Taking the refracting power of the eye as equal to 58.64 dptr., show that the size of the image on the retina will be 0.00495 mm.

7. What is the magnifying power of a convex lens of focal length 5 cm.? Ans. 5.

8. A myope of 10 dptr. uses a convex lens of focal length 5 cm. as a magnifying glass. Find the individual magnifying power, neglecting the distance of the eye from the glass. Ans. $7\frac{1}{2}$.

9. In the preceding example, what will be the individual magnifying power of the same glass in the case of an hypermetrope of 10 dptr.? Ans. $2\frac{1}{2}$.

10. A certain person cannot see distinctly objects which are nearer his eye than 20 cm. or farther than 60 cm. Within what limits of distance from his eye must a concave mirror of focal length 15 cm. be placed in order that he may be able to focus sharply the image of his eye as seen in the mirror?

Ans. In order to see a real image of his eye, the distance of the mirror must be between 43.23 cm. and 78.54 cm.; in order to see a virtual image, the distance of the mirror must be between 6.97 cm. and 11.46 cm.

11. The magnifying power of a telescope 12 inches long is equal to 8: determine the focal lengths of object-glass and eye-glass (1) when it is an astronomical telescope and (2) when it is a GALILEO's telescope.

Ans. (1) $f_1 = +10\frac{2}{3}$, $f_2 = +1\frac{1}{3}$ inches; (2) $f_1 = +13\frac{5}{7}$, $f_2 = -1\frac{5}{7}$ inches.

12. The focal lengths of the object-glass and eye-glass of an astronomical telescope are f_1 and f_2, and their diameters are $2h_1$ and $2h_2$, respectively: Show that the radius of the stop which will cut off the "ragged edge" (§ 138) is equal to

$$\frac{f_1 h_2 - f_2 h_1}{f_1 + f_2}.$$

13. A telescope is pointed at an infinitely distant object, and the eye-piece is focused so that the image is formed at the distance l of distinct vision of the eye. If the distance of the eye from the eye-piece is neglected, show that the magnifying power is $M = -f_1(l+f_2)/l.f_2$, where f_1, f_2 denote the focal lengths of the object-glass and eye-glass.

14. A RAMSDEN ocular consists of two thin convex lenses each of focal length a separated by an interval equal to $2a/3$. Show that the magnifying power of an astronomical telescope furnished with a RAMSDEN ocular is $4f_1/3a$, where f_1 denotes the focal length of the object-glass.

15. The object-glass of an astronomical telescope has a focal length of 50 inches, and the focal length of each lens of the RAMSDEN ocular is 2 inches. The distance between the two lenses in the ocular is $\frac{4}{3}$ inch. Show that the distance between the object-glass and the first lens of the ocular is 50.5 inches, and that the magnifying power is equal to $\frac{100}{3}$.

16. If a GALILEO's telescope is in normal adjustment, show that the angular diameter of the field of the image as measured at the vertex of the concave eye-glass is $2\tan\gamma' = -2h_1/(f_1+f_2)$, where h_1 denotes the radius of the object-glass and f_1, f_2 denote the focal lengths of object-glass and eye-glass.

17. The focal length of the object-glass and eye-glass of an astronomical telescope are 36 and 9 inches, respectively. If the object is infinitely distant and if the eye is placed in the eye-ring at a distance of 9 inches from the image, show that the magnifying power is equal to 3.

18. The magnifying power of a simple astronomical telescope in normal adjustment is M, and the focal length of the object-glass is f_1. Show that if the eye-glass is pushed in a distance x and the eye placed in the eye-ring, the magnifying power will be diminished by $x.M/f_1$.

19. An astronomical telescope is pointed towards the sun, and a real image of the sun is obtained on a screen placed

beyond the eye-lens at a distance d from it. If the diameter of this image is denoted by $2b$, and if the apparent diameter of the sun is denoted by 2θ, show that the magnifying power of the telescope is $M = b.\cot\theta/d$.

20. The eye is placed at a distance c from the eye-glass of a GALILEO's telescope in normal adjustment. The length of the telescope as measured from the object-glass to the eye-glass is denoted by d, the radius of the object-glass is denoted by h_1, and the radius of the pupil of the eye is denoted by g (it being assumed that g is less than the radius of the eye-glass). Show that the semi-angular diameters of the three portions of the field of view on the image-side are given by the following expressions:

$$\tan\gamma_1' = -\frac{h_1 - g.M}{d + c.M}, \quad \tan\gamma' = -\frac{h_1}{d + c.M}, \quad \tan\gamma_2' = -\frac{h_1 + g.M}{d + c.M},$$

where M denotes the magnifying power of the telescope.

CHAPTER XIV

DISPERSION AND ACHROMATISM

160. Dispersion by a Prism.—When a beam of sunlight is admitted into a dark chamber through a small circular hole A (Fig. 204) in the window shutter, a round spot of white light will be formed on a vertical wall or screen opposite the window, which will be, indeed, an image of the sun of the same kind as would be produced by a pinhole camera (§ 3); its

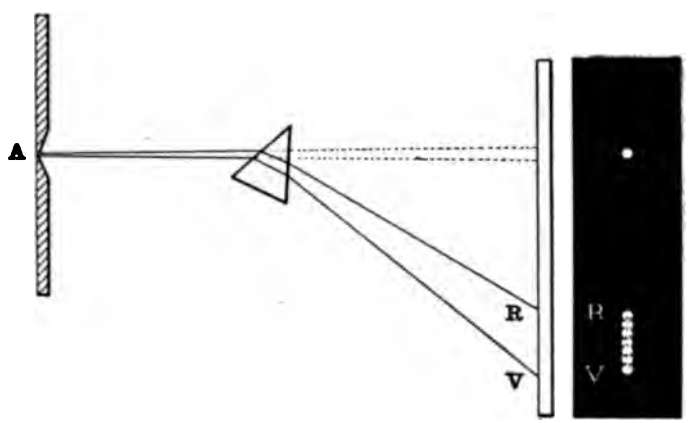


FIG. 204.—Prism dispersion: NEWTON'S experiment.

angular diameter, therefore, being equal to that of the sun, namely, about half a degree. In the track of such a beam NEWTON inserted a prism with its refracting edge horizontal and at right angles to the direction of the incident light; whereupon the white spot on the screen vanished and in its stead at a certain vertical distance above or below the place that was first illuminated there was displayed an elongated

vertical band or *spectrum*, exhibiting the colors of the rainbow in an endless variety of tints shading into each other by imperceptible gradations. This spectrum was rounded at the ends and its vertical dimension, depending on how the prism was tilted, was about 4 or 5 times as great as its horizontal dimension, the latter being equal to the diameter of the spot of white light that was formed on the screen before the interposition of the prism. For convenience of description, NEWTON distinguished seven principal or "primary" colors arranged in the following order from one end of the spectrum to the other, namely, red, orange, yellow, green, blue, indigo,* and violet; of which the violet portion of the spectrum is the longest and the orange the shortest. The red end of the spectrum was the part of the image on the screen that was least displaced by the interposition of the prism.

This phenomenon was explained by NEWTON on the assumption that ordinary sunlight is composite and consists in reality of an innumerable variety of colors all blended together; and that the index of refraction (n) of the prism, instead of having a definite value, has in fact a different value for light of each color, being greatest for violet and least for red light and varying between these limits for light of other colors.

The resolution of white light into its constituent colors by refraction is called *dispersion*. If a puff of tobacco-smoke is blown across the beam of light where it issues from the prism, only the outer parts of the beam will show any very pronounced color, because the central parts at this place will

* There has been much discussion as to what NEWTON understood by the color which he named "indigo" and which lies somewhere between the blue and the violet. Indigo, as we understand it, is more nearly an inky blue rather than a violet blue, more like green than like violet; and hence it has been suggested that NEWTON'S color vision may have been slightly abnormal. In this connection see article entitled "Newton and the Colours of the Spectrum" by Dr. R. A. HOUSTOUN, *Science Progress*, Oct. 1917.

not have been sufficiently dispersed to exhibit their individual effects. At some little distance away from the prism the entire section of the beam will be brilliantly colored.

Having pierced a small hole through the screen at that part of it where the spectrum was formed (Fig. 205), NEWTON was able by rotating the prism around an axis parallel to

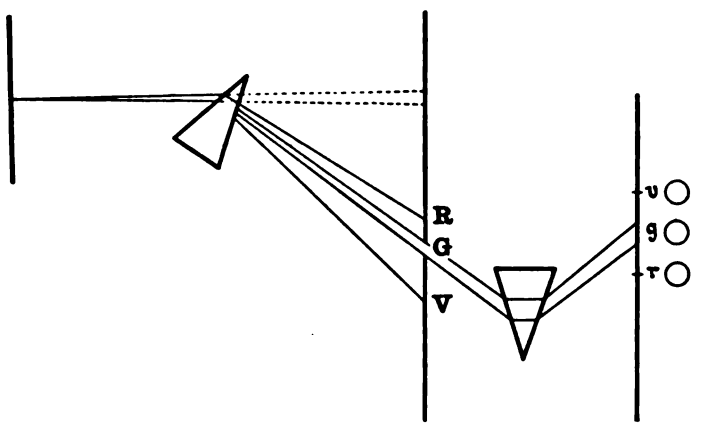


FIG. 205.—NEWTON'S experiment with two prisms; showing that light of a definite color traverses the second prism without further dispersion.

its edge to transmit rays of each color in succession through the opening to a second prism placed with its edge parallel to that of the first prism; and, agreeably to his expectations, he found that while these rays were again deviated in traversing the second prism, there was no further dispersion of the light. This experiment demonstrated that the single colors of the spectrum were irreducible or elementary and not a mixture of still simpler colors, and that the light which had been separated in this fashion from the beam of sunlight was *monochromatic light.*

If all the various components of the incident light which has been resolved by the prism are re-united again, the effect will be the same as that of the light before its dispersion.

The simplest way to achieve this result is to cause the rays to traverse a second prism precisely equal to the first, but inverted so that the dihedral angle between the planes of the adjacent faces of the two prisms is equal to 180°, the edges of the prisms being parallel. Indeed, if the two prisms were placed in contact in this way, they would form a slab of the same material throughout with a pair of plane parallel faces, for which the resultant dispersion is zero; because the colored rays would all emerge in a direction parallel to that of the incident ray which was the common path of all these

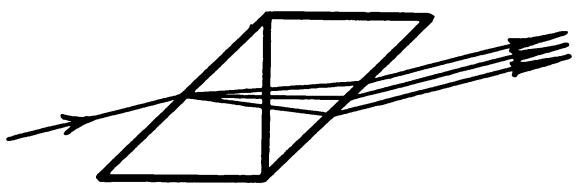


FIG. 206.—Light is not dispersed in traversing a plate with plane parallel faces surrounded by same medium on both sides.

rays before they were separated by refraction at the first face of the plate (Fig. 206).

Another and essentially different way of re-uniting the colored rays is to converge them to a single point by means of a so-called achromatic lens, as represented diagrammatically in the accompanying drawing (Fig. 207); so that the effect at the focus C where the colored rays meet is the same as that of light from the source. Beyond C the rays separate again, so that if they are received on a screen the same succession of colors will be exhibited as before, only in the reverse order. If some of the rays are intercepted before arriving at C, the color at C will be the resultant effect of the residual rays. The point B where the rays are separated on entering the prism and the point C where they are reunited by the lens are a pair of conjugate points with respect to the prism-lens system.

The solar spectrum which NEWTON obtained in his celebrated prism-experiments, described in 1672, had one serious defect, due to the fact that the colors in it were not in reality pure but consisted of a blending of two or more simple colors. When the light passes through a round hole before falling on the prism, the spectrum on the screen will be composed of a series of colored disks, each one overlapping the one next to it. The colors, therefore, are partly superposed on each other, and the eye is so constituted with respect to color vision that it cannot distinguish the separate effects and

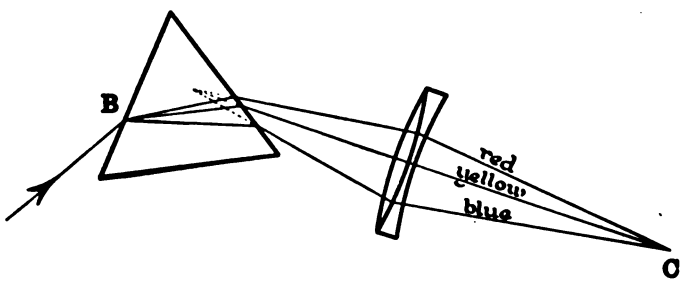


FIG. 207.—Achromatic lens used to re-unite the colored light after it has been dispersed by prism.

analyze them but obtains only a general resultant impression of the whole.

WOLLASTON'S experiments in 1802 differed essentially from NEWTON'S only in the form and dimensions of the beam of sunlight that was dispersed by the prism, but this simple modification represented a distinct advance in the mode of investigation of the spectrum. WOLLASTON admitted the sunlight through a narrow slit * whose length was parallel to

* Dr. HOUSTOUN, in the article already referred to, calls attention to the fact that in some of his prism-experiments NEWTON also employed an opening in the form of a narrow slit, and was aware of its advantages with respect to the purity of the spectrum; for NEWTON states that "instead of the circular hole," "it is better to substitute an oblong hole shaped like a long Parallelogram with its length Parallel to the Prism. For if this hole be an Inch or two long, and but a tenth

the prism-edge; and in order to diminish still more the divergence of the incident beam, a screen with a second slit parallel to the first was interposed in front of the prism, as represented in the accompanying diagram (Fig. 208). The spectrum formed in this way is far purer than that obtained with a round opening in the shutter. But a difficulty that

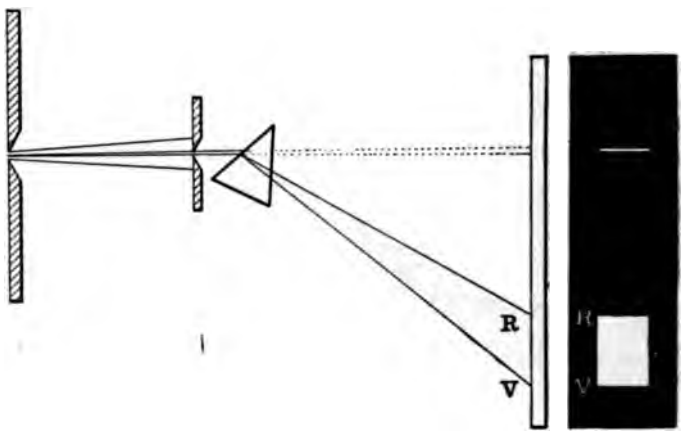


FIG. 208.—Pure spectrum obtained by causing sunlight to pass through two narrow slits before traversing prism.

inheres in both methods arises from the fact that the image formed by a prism is always virtual, and therefore a homocentric bundle of monochromatic divergent rays will necessarily be divergent after traversing a prism, so that if they are received on a screen they will illuminate a certain area on it which is the cross-section of the ray-bundle and not in any strict sense an optical image of the original source.

or twentieth part of an Inch broad or narrower; the Light of the Image, or spectrum, will be as Simple as before or simpler, and the Image will become much broader, and therefore more fit to have Experiments tried in its Light than before." The fact that NEWTON did not discover the FRAUNHOFER lines of the solar spectrum (§ 161) is probably to be explained on the supposition that his prisms were of an inferior quality of glass and that possibly also the surfaces were not as highly *polished* as they might have been.

Consequently, if the source sends out light of different colors, the effect on the screen will correspond to the sections of all the bundles of colored rays, and since these sections will overlap each other to a greater or less extent, the spectrum will not be pure. The narrower the apertures of the bundles of rays and the farther the screen is from the prism, the less

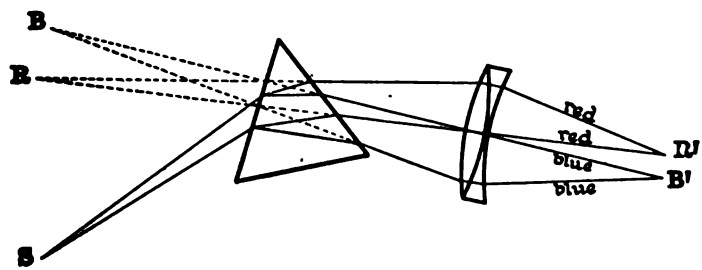


FIG. 209.—Pure spectrum obtained by slit, prism and achromatic lens.

will be the overlapping of the adjacent colors, and therefore the purer the spectrum; but on the other hand, the less also will be the illumination.

A much more satisfactory method consists in making these divergent bundles of rays convergent by means of an achromatic convex lens, as represented in Fig. 209; whereby the blue rays proceeding apparently from a virtual focus at B are brought to a real focus on the screen at B′, and, similarly, the red rays are united at R′. The plane of the diagram represents a principal section of the prism. The light originates in a luminous line or narrow illuminated slit at S parallel to the prism-edge, and the spectrum R′B′ on the screen consists of a series of colored images of this slit and is approximately pure, except in so far as the slit must necessarily have a certain width. Moreover, in the case of a very narrow slit, there are certain so-called diffraction-effects (§ 7) which are indeed of very great importance in any thorough scientific discussion of the condition of the purity of the spectrum.

161. Dark Lines of the Solar Spectrum.—WOLLASTON himself observed that the spectrum of sunlight was not absolutely continuous, but that there were certain narrow gaps or dark bands in it parallel to the slit. FRAUNHOFER (1787–1826), with his rare acumen and experimental skill, was able to obtain spectra of far higher purity than any of his predecessors, and he discovered, independently, that the solar spectrum was crossed by a very great number of dark lines, the so-called FRAUNHOFER lines, from which he argued that sunlight was deficient in light of certain colors. FRAUNHOFER counted more than 600 of these lines, but there are now known to be several thousand. One great advantage of this remarkable discovery, which FRAUNHOFER was quick to realize, consists in the fact that these lines are especially suitable and convenient for enabling us to specify particular regions or colors of the spectrum, because each of them corresponds to a certain degree of refrangibility, that is, to a perfectly definite color of light. An explanation of the origin of the dark lines of the solar spectrum may be found in treatises on physics and physical optics.

The dark lines are distributed very irregularly over the entire extent of the solar spectrum. In some cases they are sharp and fine and isolated; some of them are exceedingly close together so as to be hardly distinguishable apart; others again are quite broad and distinct. In order to describe their positions with respect to each other, FRAUNHOFER selected eight prominent lines distributed in the different regions of the spectrum, which he designated by the capital letters A (dark red), B (bright red), C (orange), D (yellow), E (green), F (dark blue), G (indigo), and H (violet). This notation is still in use, and has since been extended beyond the limits of the visible spectrum.

162. Relation between the Color of the Light and the Frequency of Vibration of the Light-Waves.—According to the undulatory theory of light, a luminous body sets up disturbances or "vibrations" in the ether which are prop-

agated in waves in all directions with prodigious velocities. The velocity of light in the free ether is about 300 million meters per second. When a train of light-waves traverses a rectilinear row of ether-particles all lying in the same medium, the distance between one particle and the nearest one to it that is in precisely the same phase of vibration is called the wave-length; and the number of waves which pass a given point in one second or the *frequency* of the undulation will be equal to the velocity of propagation of the wave divided by the wave-length. The reciprocal of the frequency will be the time taken by a single wave in passing a given point, which is called the *period* of the vibration. If the wave-length is denoted by λ, the velocity of propagation by v, the frequency by N, and the period by $T = 1/N$, the relations between these magnitudes is expressed as follows:

$$\lambda = v/N = v.T.$$

When ether-waves fall on the retina of the eye, they may excite a sensation of light provided their frequencies are neither too small nor too great, the limits of visibility being confined to waves whose frequencies lie between about 392 and 757 billions of vibrations per second. Just as the pitch of a musical note is determined by its frequency, so also the sensation which we call color appears to be more or less inexplicably associated with the frequency of the vibrations of the luminiferous ether; so that to each frequency between the limits named there corresponds a perfectly definite kind of light or color. Absolutely monochromatic light due to ether-waves of one single frequency of vibration is difficult to obtain. In general, the light which is emitted by a luminous body is more or less complex, and the sensation which it produces in the eye is due to a variety of impulses. The yellow light which is characteristic of the flame of a BUNSEN burner when a trace of common salt is burned in it is a sensation excited by the impact of two kinds of ether-waves corresponding to the double D-line of the solar spectrum

which have frequencies of about 509 and 511 billions of vibrations per second. Red light corresponds to the lowest and violet light to the highest frequency.

It is known that the velocity of light of a given color depends on the medium in which the light is propagated; and it has also been established that the velocity of light in a given medium depends on the color of the light. However, apparently light of all colors is transmitted with equal velocities *in vacuo;* and also in air, on account of its slight dispersion, there is practically no difference in the velocity of propagation of light of different colors.*

One reason for inferring that the frequency of the ether-vibrations is the physical explanation of the phenomenon of

* "When white light enters a transparent medium, the long red waves forge ahead of the green ones, which in their turn get ahead of the blue. If we imagine an instantaneous flash of white light traversing a refracting medium, we must conceive it as drawn out into a sort of linear spectrum in the medium, that is, the red waves lead the train, the orange, yellow, green, blue, and violet following in succession. The length of this train will increase with the length of the medium traversed. On emerging again into the free ether the train will move on without any further alteration of its length.

"We can form some idea of the actual magnitudes involved in the following way. Suppose we have a block of perfectly transparent glass (of ref. index 1.52) twelve miles in thickness. Red light will traverse it in 1/10000 of a second, and on emerging will be about 1.8 miles in advance of the blue light which entered at the same time. If white light were to traverse this mass of glass, the time elapsing between the arrival of the first red and the first blue light at the eye will be less than 1/6000 of a second. MICHELSON's determination of the velocity of light in carbon bisulphide showed that the red rays gained on the blue in their transit through the tube of liquid. The absence of any change of color in the variable star Algol furnished direct evidence that the blue and red rays traverse space with same velocity. In this case the distance is so vast, and the time of transit so long, that the white light coming from the star during one of its periodic increases in brilliancy would arrive at the earth with its red component so far in advance of the blue that the fact could easily be established by the spectrophotometer or even by the eye."—R. W. WOOD: *Physical Optics*, Second Edition (New York, 1911), page 101.

color is found in the fact that the color of monochromatic light remains unaltered when the light passes from one medium into another; and since the vibrations in the second medium are excited and forced by those in the first medium, it is natural to suppose that the vibration-frequency is the same in both media.

Accordingly, it is the ratio

$$\frac{v}{\lambda}=N$$

that remains constant in the transmission of monochromatic light through different media. And hence if the velocities of light in two media are denoted by v, v', and if the wave-lengths in these two media are denoted by λ, λ', then $v/\lambda=v'/\lambda'$ or $\lambda/\lambda'=v/v'$; that is, *the wave-length of light of a given color varies from medium to medium, and is proportional to the velocity of propagation of light of that color in the medium in question.* Thus, the wave-length of yellow light is shorter in glass than it is in air, because light travels more slowly in glass than in air.

Generally, therefore, when we speak of the wave-length of a given kind of light, we mean its wave-length measured *in vacuo.* The lengths of waves of light are all relatively very short, the longest, corresponding to the extreme red end of the spectrum, being less than one 13-thousandth of a centimeter, and the shortest, belonging to the extreme violet end of the visible spectrum, being less than one 25-thousandth of a centimeter. These magnitudes are usually expressed in terms of a special unit called a "tenth-meter" which is one 10-billionth part of a meter (10^{-10} meter) or in terms of a "micromillimeter" which is equal to the millionth part of a millimeter and for which the symbol $\mu\mu$ is employed ($1\mu\mu=10^{-6}$ mm.). Thus, the wave-lengths of light corresponding to the red and violet ends of the visible spectrum are about $767\mu\mu$ and $397\mu\mu$, respectively. The FRAUNHOFER line A is a broad, indistinct line at the beginning of the red part of the spectrum, wave-length 759.4 $\mu\mu$; the B-line

in the red part corresponds to light of wave-length 686.7$\mu\mu$; the C-line in the orange corresponds to light of wave-length 656.3$\mu\mu$; the D-line in the yellow is a double line, corresponding to light of wave-lengths 589.6$\mu\mu$ and 589.0$\mu\mu$; the E-line in the green corresponds to light of wave-length 527.0$\mu\mu$; the F-line in the blue corresponds to light of wave-length 486.1$\mu\mu$; the G-line in the indigo corresponds to light of wave-length 430.8$\mu\mu$; and the H-line, consisting of two broad lines in the violet, corresponds to light of wave-lengths 396.8$\mu\mu$ and 393.3$\mu\mu$.

163. Index of Refraction as a Function of the Wave-Length.—Now according to the wave-theory of light, the absolute index of refraction (n) of a medium for light of a definite color is equal to the ratio of the velocity of light *in vacuo* (V) to its velocity (v) in the medium in question (§ 33); that is,

$$n = \frac{V}{v}.$$

Strictly speaking, therefore, the index of refraction of a medium, without further qualification, is a perfectly vague expression, because each medium has as many indices of refraction as there are different kinds of monochromatic light. When the term is used by itself, it is generally understood to mean the index of refraction corresponding to the D-line in the bright yellow part of the solar spectrum, which is characteristic of the light of incandescent sodium vapor. Hence,

$$n_D = \frac{\text{velocity of yellow light } \textit{in vacuo}}{\text{velocity of yellow light in the medium in question}}$$

$$= \frac{\text{wave-length of yellow light } \textit{in vacuo}}{\text{wave-length of yellow light in the given medium}}$$

In the following table the values of the indices of refraction of several transparent liquids are given for light corresponding to the FRAUNHOFER lines A, B, C, D, E, F, G, and H.

	A	B	C	D	E	F	G	H
Wave-length in $\mu\mu$	759.4	686.7	656.3	589.0	527.0	486.1	430.8	396.8
Alcohol	1.359	1.360	1.361	1.363	1.365	1.367	1.371	1.374
Benzene	1.493	1.495	1.497	1.503	1.507	1.514	1.524	1.536
Sulphuric Acid	1.610	1.616	1.620	1.629	1.642	1.654	1.670	1.702
Water	1.329	1.331	1.332	1.334	1.336	1.338	1.341	1.344

It may be remarked that, in general, the shorter the wave-length, the greater will be the index of refraction of a substance. But the exact relation between the index of refraction and the wave-length of the light has to be determined empirically for each substance. There is, indeed, a certain group of substances which form an exception to the general statement made above, and which yield refraction-spectra with the order of the colors partially or entirely reversed. This phenomenon is called *anomalous dispersion*.

164. Irrationality of Dispersion.—Other things being equal, the length of the spectrum or the interval between

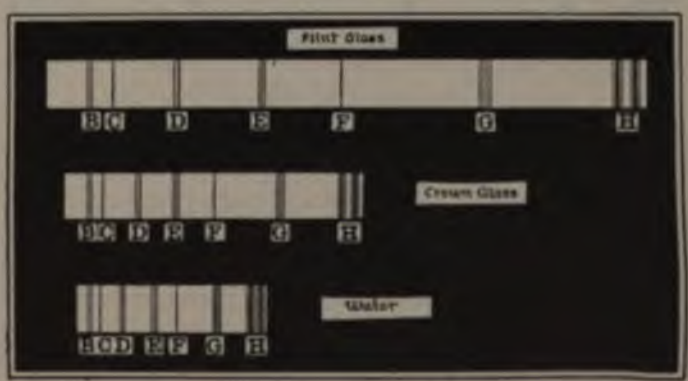


FIG. 210.—Irrationality of dispersion.

a given pair of FRAUNHOFER lines depends essentially on the nature of the refracting medium, so that, in general, as shown by the table in the preceding section, the dispersion of two colors will be found to be different for different substances.

For example, the dispersion of glass is greater than that of water, and the dispersion of so-called flint glass is higher that of so-called crown glass. In Fig. 210 are exhibited the relative lengths of the different regions of the solar spectra cast on the same screen under precisely the same circumstances by prisms of equal refracting angles made of water, crown glass and flint glass. The length of the spectrum may be increased by shifting the screen farther from

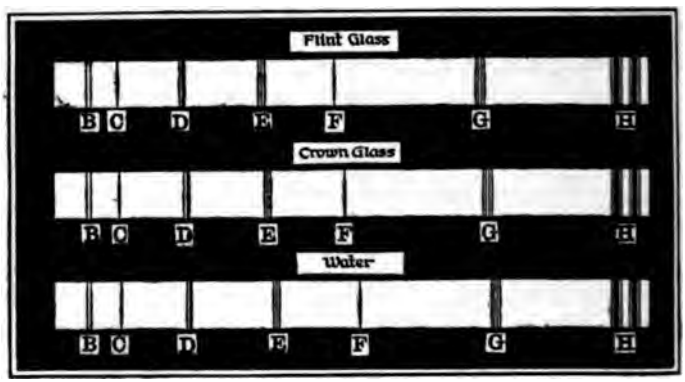


FIG. 211.—Irrationality of dispersion.

the prism, and Fig. 211 shows the relative positions of the FRAUNHOFER lines B, C, D, E, F, G and H, when the lengths of the spectra of the crown glass prism and the water prism have been elongated in this manner until their lengths are both equal to the length of the spectrum of the flint glass prism for the interval between the FRAUNHOFER lines B and H. The other lines in the three spectra do not coincide at all. Moreover, it appears that the dispersion of water for the colors towards the red end of the spectrum is relatively high, whereas the dispersion of the flint glass is relatively high towards the blue end. In the spectrum of flint glass the interval between G and H, and in the spectrum of water the interval between B and F, is greater than it is in either of the other spectra. If the law of the variation of the index of re-

fraction with the color of the light has been found empirically for one substance, this will not afford any clue to the corresponding law in the case of another substance. Diamond, for example, is very highly refracting but shows comparatively little dispersion, whereas flint glass which has a much lower index of refraction gives a much higher dispersion; on the other hand, fluorite has a low index of refraction and at the same time a low dispersion. This phenomenon which is characteristic of refraction-spectra is known as the *irrationality of dispersion.*

165. Dispersive Power of a Medium.—In the case of a prism of small refracting angle β the deviation is given by the formula $\epsilon=(n-1)\beta$, as was explained in § 60. Let the letters P and Q be used to designate two colors, and let n_P and n_Q denote the indices of refraction of the prism-substance for these colors. If the angles of deviation are denoted by ϵ_P and ϵ_Q then $\epsilon_Q-\epsilon_P=(n_Q-n_P)\beta$, and, consequently, for a thin prism the angular magnitude of the interval in the spectrum between the colors P and Q is proportional to the difference of the values of the indices of refraction. This difference (n_Q-n_P) is called the *partial dispersion* of the substance for the spectrum-interval P, Q. Thus, in the brightest part of the spectrum comprised between the FRAUNHOFER lines C and F, the partial dispersion is (n_F-n_C). The deviation of a prism of small refracting angle β for light corresponding to the D-line which lies between C and F is $\epsilon_D=(n_D-1)\beta$, and since $\epsilon_F-\epsilon_C=(n_F-n_C)\beta$, we obtain:

$$\frac{\epsilon_F-\epsilon_C}{\epsilon_D}=\frac{n_F-n_C}{n_D-1}.$$

This ratio of the angular dispersion of two colors to their mean dispersion is called the *dispersive power* or the *relative dispersion* of the substance for the two colors, which are usually red (C) and blue (F); so that the dispersive power of an optical medium with respect to the visible spectrum may be defined to be the quotient of the difference (n_F-n_C)

between the indices of refraction for red and blue light by $(n_D - 1)$, where n_D denotes the index of refraction for yellow light. The values of the dispersive powers of the various kinds of optical glass that are of chief practical importance in the construction of optical instruments vary from about $\frac{1}{65}$ to about $\frac{1}{34}$; although there are compositions of glass with values of the dispersive power not comprised within these limits. Instead of assigning the value of the dispersive power of a substance, it is more convenient to adopt ABBE's method and employ the reciprocal of this function, which is denoted by the Greek letter ν, and which is known, therefore, as the ν-value of the substance; thus,

$$\nu = \frac{n_D - 1}{n_F - n_C}.$$

If the ν-value of one substance is less than that of another, the dispersive power of the former will be correspondingly greater than that of the latter.

It is this constant ν that is the essential factor to be considered in the selection of different kinds of glass suitable to be used in making a so-called achromatic combination of lenses or prisms. Curiously enough, NEWTON persisted in maintaining that the dispersion of a substance was proportional to the refraction, which is equivalent to saying that the dispersive powers of all optical media are equal; and, consequently, he despaired of constructing an achromatic combination of lenses which would refract the rays without at the same time dispersing the constituent colors. This condition, however, is an essential requirement in the object-glass of a telescope, and it was just because NEWTON and his followers believed that a lens of this kind was in the nature of things unattainable that they expended their efforts in the direction of perfecting the reflecting telescope in which the convex lens was replaced by a concave mirror. On the other hand, from the assumption that the optical system of the human eye is free from color-faults (which is by no means *true*), it was argued, notably by JAMES GREGORY in England

(about 1670) and long afterwards by EULER in Germany (1747), that NEWTON's conclusions as to the impossibility of an achromatic combination of refracting media were erroneous. In fact, an English gentleman named HALL succeeded in 1733 in constructing telescopes which yielded images free from serious color faults. KLINGENSTIERNA in Sweden in 1754 demonstrated the feasibility of combining a pair of prisms of different kinds of glass and of different refracting angles so as to obtain, in one case, deviation without dispersion and, in another case, dispersion without deviation.

But in its practical results the most important advance along this line was achieved by the painstaking and original work of the English optician JOHN DOLLOND. Impressed by the force of KLINGENSTIERNA's demonstration, he carefully repeated NEWTON's crucial experiment in which a glass prism was inclosed in a water prism of variable refracting angle; and having found that the results of this experiment were exactly contrary to those stated by NEWTON, he was led also to the opposite conclusion. After much perseverance DOLLOND had succeeded by 1757 in making achromatic combinations of several different types, which produced a more or less colorless image of a point-source on the axis of the system. In its original form the combination consisted of a double convex "crown glass" lens cemented to a double concave "flint glass" lens. As a rule, the focus of the blue rays will be nearer a convex lens and farther from a concave lens than the focus of the red rays; and hence by combining a convex crown glass lens of relatively lower refractive index (shorter focus) and less dispersive power with a concave flint glass lens of higher refractive index and higher dispersive power, a resultant system may be obtained which still has a certain finite focal length and in which at the same time the opposed color-dispersions for two colors, say, red and blue, are compensated.

166. Optical Glass.—NEWTON's error in supposing that for all substances the dispersion was proportional to the index

of refraction retarded the development of technical optics for a long time to come. Although DOLLOND's achievement, mentioned above, was one of far-reaching importance for the practical construction of optical instruments, the great difficulty in the way of utilizing and applying the principle was to be found in the fact that the actual varieties of optical glass at the disposal of the optician were exceedingly limited in number; although from time to time systematic efforts were made, notably by FRAUNHOFER (about 1812) in Germany and by FARADAY (1824), HARCOURT (1834) and STOKES (about 1870) in England, to remedy this deficiency, by discovering and manufacturing new compositions of glass suitable for optical purposes. For a long time after FRAUNHOFER's epoch the art of making optical glass was confined almost exclusively to France and England. It was a fortunate coincidence that just about the time when E. ABBE had reached the conclusion that no further progress in optical construction could be expected unless totally new varieties of optical glass were forthcoming, O. SCHOTT was already beginning to experiment with new chemical combinations and processes of manufacture in his glass works at Jena. Thanks to the systematic and indefatigable efforts of these two collaborators, who were also encouraged by the Prussian government, the obstacle which had stood so long in the way of the improvement and development of optical instruments was at length triumphantly overcome by the successful production of an entire new series of varieties of optical glass with properties in some instances almost beyond the highest expectations. The first catalogue of the *Glastechnisches Laboratorium* at Jena was issued in 1885; which marked the beginning of the manufacture of the renowned Jena glass, to which more than to any other single factor the remarkable development of modern optical instruments is due. From that time to the present the great province of applied optics may almost be said to have become a German territory.

The earlier so-called "ordinary" varieties of optical

glass were silicates in which the basic constituents were lime (crown glass) or lead (flint glass) combined with soda (Na_2CO_3) or potash (K_2CO_3) or both. The newer kinds of optical glass have been produced by employing a much greater variety of chemical substances, including, in addition to those named above, hydrated oxide of aluminum (Al_2O_3,H_2O), barium nitrate (BaN_2O_6), zinc oxide (ZnO), etc., and boric acid (H_3BO_3) or phosphoric acid which to a greater or less extent replace the silica (SiO_2) in the older types. Some of the new compounds have been found to have slight durability, and for this and other reasons certain products formerly listed in the Jena glass catalogue have been discontinued. At present, besides the old "ordinary" silicate crown and flint, the chief varieties are barium and zinc silicate crown, boro-silicate crown, dense baryta crown, baryta flint, antimony flint, borate glass and phosphate glass. The table on the following page contains a list of certain varieties of Jena glass arranged in the order of their ν-values. In the Jena glass catalogue the values of the dispersion are given also for the spectrum-intervals $n_D-n_{A'}$, n_F-n_D, $n_{G'}-n_F$ (where A′ and G′ are the lines corresponding to the wave-lengths 768 and 434$\mu\mu$, respectively), together with the values of the so-called relative partial dispersions obtained by dividing each of these numbers by the value of (n_F-n_C).

It has recently been proposed to describe an optical glass by means of two numbers of 3 digits each, separated by an oblique line. The first number gives the first three figures after the decimal point in the value of n_D, while the second number is equal to 10 times the value of ν. Thus, for example, the second glass in the table would be described as crown glass No. 559/669.

SELECTED VARIETIES OF JENA GLASS

Description	Index of Refraction n_D	Mean Dispersion $n_F - n_C$	$\frac{n_D - 1}{n_F - n_C} = \nu$
Light phosphate crown	1.5159	0.007 37	70.0
Medium phosphate crown	1.5590	0.008 35	66.9
Boro-silicate crown	1.5141	0.008 02	64.1
Boro-silicate crown	1.5103	0.008 05	63.4
Silicate crown	1.5191	0.008 60	60.4
Silicate crown	1.5215	0.008 75	59.6
Silicate crown	1.5127	0.008 97	57.2
Densest baryta crown	1.6112	0.010 68	57.2
Barium crown	1.5726	0.009 95	57.5
Dense baryta crown	1.6130	0.010 87	56.4
Dense baryta crown	1.6120	0.010 98	55.7
Baryta flint	1.5664	0.010 21	55.5
Borate flint	1.5503	0.009 96	55.2
Baryta flint	1.5489	0.010 25	53.6
Baryta flint	1.5848	0.011 04	53.0
Antimony flint	1.5286	0.010 25	51.6
Boro-silicate flint	1.5503	0.011 14	49.4
Extra light flint	1.5398	0.011 42	47.3
Baryta flint	1.5825	0.012 55	46.4
Ordinary light flint	1.5660	0.013 19	42.9
Silicate flint	1.5794	0.014 09	41.1
Baryta flint	1.6235	0.015 99	39.1
Heavy borate flint	1.6797	0.017 87	38.0
Silicate flint	1.6138	0.016 64	36.9
Silicate flint	1.6489	0.019 19	33.8
Dense silicate flint	1.7174	0.024 34	29.5
Densest silicate flint	1.9626	0.048 82	19.7

In recent years in France, England and the United States much attention has been bestowed on the study of the composition and manufacture of optical glass, and according to the 1916–17 report of the British Committee of the Privy Council for Scientific and Industrial Research (summarized in *Nature*, Vol. 100, pp. 17–20), Professor JACKSON in England "has succeeded in defining the composition of the bath mixtures necessary for the production of several glasses hitherto manufactured exclusively in Jena, including the famous fluor-crown glass," and, moreover, "he has also discovered three completely new glasses with properties hitherto unobtainable." However, it seems improbable

that any essential changes in the optical properties of glass are to be obtained by the use of materials that have not already been tried. The index of refraction of all glasses at present available are comprised between 1.45 and 1.96. The mineral fluorite (calcium fluoride), which is used in the best modern microscope objectives, has an index of refraction of 1.4338 and a ν-value of 95.4, so that in both respects it lies beyond the limits attainable with glass. Other crystalline transparent minerals, notably rock crystal or quartz, have already been employed in lens-systems, and any essential improvement in the range of optical instruments in the future is more likely to come from an adaptation of these mineral substances than from the production of new kinds of glass.

The difficulties involved in the manufacture of high-grade optical glass are very great, and the utmost care has to be exercised throughout every stage of the process. Not only must the raw materials themselves be free from impurities as far as possible, but the physical and chemical nature of the fireclays used in the pots or crucibles also requires the most painstaking care and preparation. The empty crucible is dried slowly and then heated gradually for several days until it comes to a bright red glow. Fragments of glass left over from a previous melting and of the same chemical composition as the glass which is in process of making are introduced into the pot and melted. The raw materials, pulverized and mixed in definite proportions, are placed in the pot in layers little by little at a time, and the pot, which is covered to protect the contents from the furnace gases is maintained at a sufficiently high temperature (between about 800 and 1000° C.) until the contents are all melted together. The molten mass is usually full of bubbles of all sizes, and the temperature must be raised until these are all gotten rid of as far as possible. This entire process takes a longer or shorter time depending on circumstances, say, from 24 to 36 hours or more. After skimming off the impurities on the

surface, the mixture is allowed to cool gradually, and at the same time it is kept constantly stirred in order to make the glass as homogeneous as possible. This part of the process requires constant care. When the glass in cooling has become quite viscous, so that it is no longer possible to continue the stirring, it is allowed to cool very slowly over a period of days or even weeks. Usually at the end of the cooling process the solid contents of the pot will be found to be broken into irregular fragments of optical glass in the first stage of its manufacture. These fragments are carefully examined to see whether they are homogeneous and above all free from striæ; but the broken surfaces are so irregular that this preliminary examination is necessarily very imperfect. The pieces which pass muster in this way are selected for molding and annealing. The lumps of glass are placed in suitable molds made of iron or fireclay and heated until the glass becomes soft like wax, so that it takes the form of the mold usually with the aid of external pressure. The molded pieces are then annealed by being cooled gradually for a week or longer. They are in the form of disks or rectangular blocks of approximately the right size for being made into lenses and prisms. At this stage the glass has to be subjected to the most rigid testing to see if it is really suitable for optical purposes. Two opposite faces on the narrow sides are ground flat and parallel and polished so that the slab can be inspected in the direction of its greatest diameter. If any striæ or other imperfections are found, the piece will have to be rejected and melted over again. Even in case there are no directly visible defects, there may be internal strains which will be revealed by examination with polarized light. Slight strains are not always serious, but even these will impair the image in a large prism or lens. These strains can be gotten rid of by heating the glass to a temperature between 350 and 480° C., depending on the composition, and then cooling very slowly and uniformly over a period of about six weeks. It is very difficult to obtain pieces of op-

tical glass which do not contain minute bubbles, and indeed they are often to be found in the best kinds of glass.

Of course, the process as above described varies in details according to the special nature of the glass, but enough has been said to enable the reader to form some idea of the patience and skill which are required in the manufacture of optical glass. A yield of 20 per cent. of the total quantity of glass melted is considered good. The glass to be used for photographic lenses has to fulfill the most exact requirements and must be of the highest quality.

167. Chromatic Aberration and Achromatism.—Since the index of refraction varies with the color of the light, and since this function enters in one form or another in all optical calculations, it is obvious, for example, that the positions of the cardinal points of a lens-system will, in general, be different for light of different colors; and that there will be a whole series of colored images of a given object depending on the nature of the light which it radiates, these images being all more or less separated from each other and of varying sizes. This phenomenon is called *chromatic aberration*, and unless it is at least partially corrected, the definition of the resultant image is very seriously impaired. In an optical system which was absolutely free from chromatic aberration all these colored images would coalesce into a single composite image which, so far as the quality of the light was concerned, would be a faithful reproduction of the object. But nothing at all comparable to this ideal condition of *achromatism* can be achieved in the case of any actual lens-system. In fact, the term achromatism by itself and without any further explanation is entirely vague, for an optical system may be achromatic in one sense without being at all so in other senses. For example, the images corresponding to different colors may all be formed in the same plane and yet be of different sizes, or *vice versa*. Fortunately, however, the fact that it is impossible to achieve at best more than a partial achromatism is not such a serious matter after all. The kind of achromat-

ism which is adapted for one type of optical instrument may be entirely unsuited to another type. Thus, it is absolutely essential that the colored images formed by the object-glass of a telescope or microscope shall be produced as nearly as possible at one and the same place (achromatism with respect to the location of the image), whereas, since the images in this case do not extend far from the axis, the unequal color-magnifications are comparatively unimportant. On the other hand, in the case of the ocular systems of the same instruments, the main consideration will be a partial achromatism with respect to the magnification or the apparent sizes of the colored images. The object-glass of a telescope must be achromatic with respect to the position of its focal point, and the ocular must be achromatic with respect to its focal length.

An optical system which produces the same definite effect for light of two different wave-lengths, no matter what that special effect may be, is to that extent an *achromatic system*. A combination which is achromatic, even in its limited sense, for a certain prescribed distance of the object will, in general, not be achromatic when the object is placed at a different distance. No lens composed of two kinds of glass only can be achromatic for light of all different colors. It can be constructed, for example, so that it will bring the red and violet rays accurately to the same focus at a prescribed point on the axis; but then the yellow, green and blue rays will, in general, all have different foci, some of which will be nearer the lens than the point of reunion of the red and violet light while others will lie farther away. Accordingly, when achromatism has been attained in the case of two chosen colors, there will usually remain an uncorrected residual dispersion or so-called *secondary spectrum*, which under certain circumstances may impair the definition of the image to such a degree as to be very injurious and annoying. It is necessary to abolish the secondary spectrum in the object-glass of a microscope. This may be done by using more than two

kinds of glass. There is also the possibility of diminishing the secondary spectrum by employing two kinds of glass whose relative partial dispersions (§ 166) are very nearly the same for all the spectrum-intervals; and, in fact, one of the principal items in the ABBE-SCHOTT programme for the manufacture of optical glass was the production of various pairs of flint and crown glass suitable for such combinations, so that the dispersions in the different regions of the spectrum should be, for each pair, as nearly as possible proportional. This purpose was satisfactorily accomplished, and we have now achromatic lenses of a far more perfect kind than could be made out of the older kinds of glass. This higher degree of achromatism is called *apochromatism*. An apochromatic photographic lens is absolutely essential in the three-color process of photography in which the three images taken through light-filters on a plate of medium or large size must be superposed as exactly as possible. In most ordinary optical systems, however, the secondary spectrum is relatively unimportant, and achromatism with respect to two principal colors will usually be found to be sufficient.

168. "Optical Achromatism" and "Actinic Achromatism."—The character and extent of the secondary spectrum (§ 167) of an achromatic combination of lenses will evidently depend essentially on the *choice of the two principal colors* for which the achromatism is to be achieved. This choice will be determined by the purpose for which the instrument is intended and the mode of using it. Thus, if the system is to be an *optical* instrument in the strict literal sense of the word, that is, if it is constructed to be used subjectively in conjunction with the eye, we shall be concerned primarily with the physiological action of the rays on the retina of the human eye; whereas in the case of a photographic lens which is used to focus an image on a prepared sensitized plate, it is important to have achromatism with respect to the so-called actinic rays corresponding to the violet and ultra-violet regions of the spectrum, because these are the rays

which are most active on the ordinary bromo-silver gelatine plate.

The retina of the human eye is most sensitive to the kind of light which is comprised within the interval between the lines C and F, with a distinct maximum of visual effect corresponding to wave-lengths lying somewhere between the lines D and E. Accordingly, in an optical instrument which is to be applied to the eye, it is usually desirable to unite the red and blue rays as nearly as possible at the focus of the yellow rays. If, for example, the system is assumed to be a convergent combination of two thin lenses in contact (as in the case of the object-glass of a telescope), it will be found that the focal points corresponding to the colors (say, green and yellow) between C and F will lie nearer the lens and the focal points corresponding to the other colors (dark red, dark blue and violet) will lie farther from it than the common focal point of the two principal colors C and F. Moreover, the residual color-error or secondary spectrum in this case will be least for some color very nearly corresponding to the D-line, which is a favorable circumstance, since, as above stated, this is the region of the brightest part of the visible spectrum. Achromatism with respect to the colors C and F $\left(\nu = \frac{n_D - 1}{n_F - n_C}\right)$ is sometimes called *optical achromatism.*

On the other hand, in the construction of a photographic lens a kind of compromise must be effected between the convergence of the visual rays and the so-called actinic rays, because the image has to be focused first on the ground glass plate by the eye and afterwards it has to be received on the sensitized plate or film which is inserted for exposure in the camera in the place of the translucent focusing screen. Accordingly, for ordinary photographic practice an exact coincidence of the "optical" and "actinic" images is demanded. Here it is found that the best results are obtained by uniting the colors corresponding to the D-line and the violet band in the spectrum of hydrogen, which, since it is

not far from the G-line, may be designated by G′ (434$\mu\mu$). This is sometimes called *actinic* or *photographic achromatism* for which the function ν has a special value, namely:

$$\bar{\nu} = \frac{n_D - 1}{n_{G'} - n_D}.$$

If the photographic lens is a combination of two thin lenses in contact, which is achromatic for the colors D and G′, the focus of the rays corresponding to the blue-green region of the spectrum will be nearer the lens than the common focus of the two principal colors and the focus of the bright red rays will be farther from the lens. In an achromat of this kind the residual dispersion will usually be quite large for both the "optical" and the "actinic" image, but for most practical purposes the definition of the image in either case is good enough. In astrophotography the focus of the camera is determined once for all, and a lens for stellar photography is usually designed to have an *entirely actinic achromatism*, the two principal colors in this case corresponding to the F-line (486$\mu\mu$) and the violet line in the mercury-spectrum (405$\mu\mu$). The rays belonging in these two colors are made to unite as nearly as possible at the focus of the rays corresponding to the G′-line, which is approximately the place of maximum actinic action. In a photographic achromat of this kind the foci of the green, yellow and red rays will lie beyond the actinic focus in the order named.

169. Achromatic Combination of Two Thin Prisms.— Two prisms of different substances may be combined so as to obtain achromatism in the sense that rays of light corresponding to a definite pair of colors will issue from the system in parallel directions, as represented in Fig. 212. When an object is viewed through the combination, the red and blue rays, for example, will be fused or superposed and the residual color-effect will be comparatively slight. By employing a greater number of prisms a more perfect union of colors could be obtained, but usually two prisms are sufficient.

The problem is simplified by assuming that the refracting angles of the prisms, denoted here by β and γ, are both small; so that the deviation produced by each prism may be considered as proportional to its refracting angle, according to the approximate formula deduced in § 60. Usually,

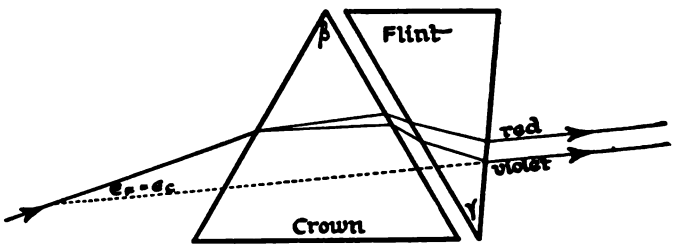


FIG. 212.—Achromatic combination of two thin prisms.

the two prisms are cemented together with their edges parallel but oppositely turned, as shown in the diagram, so that the thicker portions of one prism are adjacent to the thinner portions of the other; accordingly, the total deviation (ϵ) will be equal to the arithmetical difference of the deviations produced by each prism separately.

Let P, Q and R designate three elementary colors, the color Q being supposed to lie between P and R in the spectrum; and let the indices of refraction for these three colors be denoted by n_P', n_Q' and n_R' for the first prism and by n_P'', n_Q'' and n_R'' for the second prism. The total deviations for the three colors will be:

$$\epsilon_P = (n_P' - 1)\beta - (n_P'' - 1)\gamma, \qquad \epsilon_Q = (n_Q' - 1)\beta - (n_Q'' - 1)\gamma,$$
$$\epsilon_R = (n_R' - 1)\beta - (n_R'' - 1)\gamma.$$

Now if the system is to be achromatic with respect to the colors P and R, the condition is that $\epsilon_P = \epsilon_R$, which, therefore, is equivalent to the following:

$$\frac{\beta}{\gamma} = \frac{n_R'' - n_P''}{n_R' - n_P'};$$

that is, the refracting angles of the prisms must be inversely

proportional to the partial dispersions of the two media for the two given colors.

Moreover, the deviation of the rays of the intermediate color Q will be:

$$\epsilon_Q = (n_R' - n_P') \left\{ \frac{n_Q' - 1}{n_R' - n_P'} - \frac{n_Q'' - 1)}{n_R'' - n_P''} \right\} \beta.$$

Actually the colors P, Q and R are usually chosen to correspond to the FRAUNHOFER lines C, D and F, respectively, in which case the combination will be achromatic with respect to C (red) and F (blue). Thus, the fractions inside the large brackets are the ν-values of the two kinds of glass. Accordingly, for an achromatic combination of two thin prisms for which the deviation ϵ_D has a finite value, whereas the dispersion $(\epsilon_C - \epsilon_F)$ is abolished, we have the following formulæ:

$$\frac{\beta}{\gamma} = \frac{n_F'' - n_C''}{n_F' - n_C}, \quad \epsilon_D = (n_F' - n_C')\ (\nu' - \nu'')\beta.$$

Consider, for example, a combination of two kinds of Jena glass as follows:

	n_D	$n_F - n_C$	ν
Light Phosphate Crown	1.5159	0.007 37	70.0
Borate Flint	1.5503	0.009 96	55.2

Assuming that the angle of the crown glass prism is $\beta = 20°$, we find: $\gamma = 14.8°$, $\epsilon_D = 2.18°$. Generally speaking, those pairs of glasses in which the partial dispersions are more nearly equal will be found to be best adapted for achromatic combinations.

170. Direct Vision Combination of Two Thin Prisms.— In the case of an ordinary prism-spectroscope the rays are deflected in passing through the system, so that in order to view the spectrum the eye has to be pointed not directly towards the luminous source, but in some oblique direction; which is sometimes inconvenient, especially in astrophysical observations. Accordingly, various prism-systems have been proposed which are designed so that rays corresponding to some definite standard color are finally bent back into

their original direction, with the result that there is dispersion without deviation, which is an effect precisely opposite to that which is obtained with an achromatic prism. In these so-called *direct vision prisms* (*prismes à vision directe*)

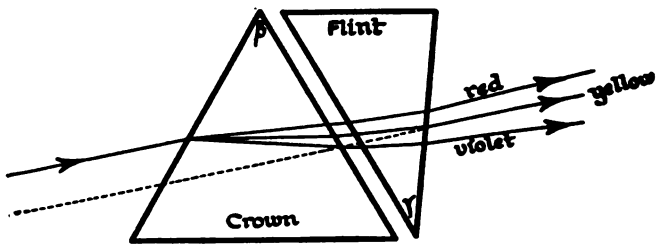


FIG. 213.—Direct vision prism combination (dispersion without deviation).

the spectrum of an illuminated slit will be seen in the same direction as the slit itself. The condition that the light corresponding, say, to the FRAUNHOFER D-line shall emerge from the system in the same direction as it entered is $\epsilon_D = 0$. Assuming that the combination is composed, as before, of two thin prisms juxtaposed in the same way (Fig. 213), and employing the same symbols (§ 169), we derive immediately the following formulæ:

$$\frac{\beta}{\gamma} = \frac{n_D'' - 1}{n_D' - 1},$$

$$\epsilon_C - \epsilon_F = (n_D' - 1)\left(\frac{1}{\nu''} - \frac{1}{\nu'}\right).$$

Consider, for example, the following combination:

	n_D	ν
Light Phosphate Crown	1.5159	70.0
Heavy Silicate Flint	1.9626	19.7

the difference of the ν-values here being very great. If we put $\beta = 20°$, we find: $\gamma = 10.72°$, $\epsilon_C - \epsilon_F = 22.56'$.

It will be profitable for the student to satisfy himself by several examples that two kinds of glass which are best adapted for a direct vision prism combination are on the

contrary not very suitable for an achromatic prism, and *vice versa;* as might naturally be expected, since the effects are opposite in the two cases. Generally speaking, the two kinds of glass used for a direct vision prism should have very different ν-values, as in the illustration given above.

In the case of prisms of large refracting angles, the formulæ here and in § 169 are hardly to be considered as even approximate.

171. Calculation of Amici Prism with Finite Angles.— Accurate formulæ for the calculation of an achromatic or direct vision prism-system may easily be derived when the

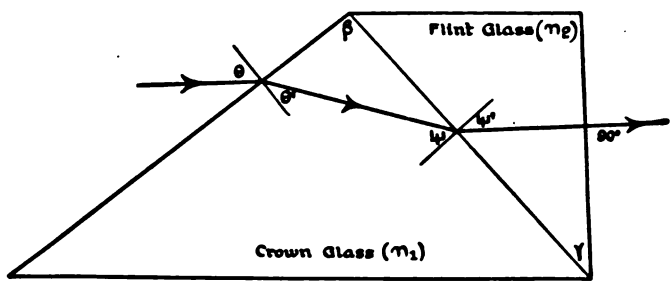


FIG. 214.—Direct vision prism combination. Diagram represents one-half of so-called AMICI direct vision prism.

system consists of only two prisms. As an illustration of the method in the case of a direct vision prism, let us employ here the symbols n_1 and n_2 to denote the indices of refraction of the crown glass prism and the flint glass prism, respectively, for light of some standard wave-length; and let β and γ denote their refracting angles. We shall suppose also that the two prisms are cemented along a common face, as represented in Fig. 214. A ray of the given wave-length is incident on the crown glass prism at an angle θ and is refracted into this medium at the angle θ', so that

$$n_1 . \sin \theta' = \sin \theta. \qquad (1)$$

If the angles of incidence and refraction at the surface of

separation of two kinds of glass are denoted by ψ and ψ', then

$$n_1.\sin\psi = n_2.\sin\psi', \qquad (2)$$

$$\theta' = \beta - \psi; \qquad (3)$$

the angles here being all reckoned as positive. If, finally, it is assumed that the ray meets the second face of the second prism normally and issues again into the air in the same direction as it had originally, then also:

$$\psi' = \gamma, \quad (4) \quad \text{and} \quad \theta = \beta - \gamma. \qquad (5)$$

The problem consists in determining the angle of one of the prisms when the angle of the other is given. Suppose, for example, that an arbitrary value is assigned to the acute angle γ, and it is required to find the magnitude of the angle β. Substituting in (1) the values of θ, θ' as given in (3) and (5), we obtain:

$$n_1.\sin(\beta - \psi) = \sin(\beta - \gamma),$$

whence we derive:

$$\tan\beta = \frac{n_1.\sin\psi - \sin\gamma}{n_1.\cos\psi - \cos\gamma}.$$

Eliminating ψ' from (2) and (4), we find:

$$n_1.\sin\psi = n_2.\sin\gamma,$$

and consequently also:

$$n_1.\cos\psi = \sqrt{n_1^2 - n_2^2.\sin^2\gamma}.$$

Accordingly, the value of β in terms of n_1, n_2 and γ is given by the formula:

$$\tan\beta = \frac{(n_2 - 1)\sin\gamma}{\sqrt{n_1^2 - n_2^2\sin^2\gamma} - \cos\gamma}.$$

If, on the other hand, the value of the angle β has been chosen arbitrarily, the calculation of γ will be found to be trigonometrically a little more difficult. It is left as an exercise for the student to show that:

$$\tan\gamma = -\frac{n_2 - 1 + \sqrt{n_1^2(n_2-1)^2 + (n_1^2 - 1)(n_2^2 - n_1^2)\tan^2\beta}}{(n_2^2 - n_1^2)\tan^2\beta + (n_2 - 1)^2}\tan\beta.$$

If it is desired that the emergent ray shall not only be parallel to the incident ray but that its path shall be along the

same straight line, it is necessary to add to the above another combination identical with it and placed so that the two flint glass prisms constitute in reality one single prism of refracting angle 2γ inserted between two equal crown glass prisms each of refracting angle β, as shown in Fig. 215; and,

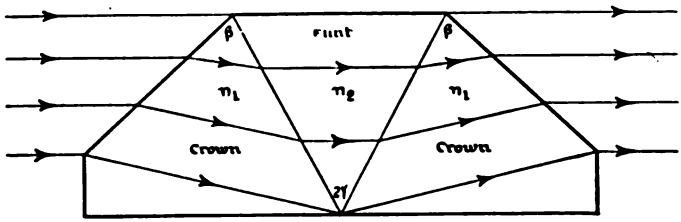


FIG. 215.—AMICI direct vision prism.

in fact, this is the actual construction of the common form of the AMICI prism. Suppose, for example, that the angle $\gamma = 45°$ and that the two kinds of glass are those described in the Jena catalogue as "light phosphate crown" and "heavy silicate flint" with indices $n_1 = 1.5159$ and $n_2 = 1.9626$ corresponding to the D-line; then we find that the angle $\beta = 98° 7.4'$.

172. Kessler Direct Vision Quadrilateral Prism.—One of the principal objections to a train of prisms is the loss of light by reflection at the various surfaces and also by absorption in traversing the successive media. Partly with a view to diminishing these losses and partly also on account of other advantages, many forms of direct vision prism have been proposed which are made of one piece of glass with four or more plane faces; in all of which, however, the principle is the same, namely, by means of a series of total internal reflections to bend the rays corresponding to some standard intermediate color back finally into their original direction. The simplest of all these devices is the four-faced prism ABCD (Fig. 216) proposed by KESSLER, a principal section of which has the form of a quadrilateral with perpendicular diagonals. The ray of standard wave-length enters the prism

and leaves it in a direction parallel to the diagonal BD; it is totally reflected twice, first at the face BA and again at the face AD, the path of the ray between these reflections being parallel to its direction at entrance and emergence. More-

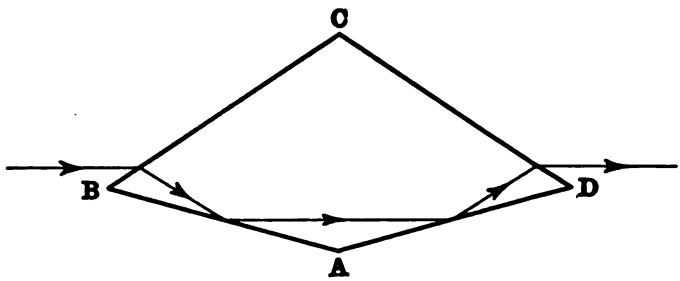


FIG. 216.—KESSLER direct vision prism.

over, in virtue of the symmetry of the prism, the path of the emergent ray will be a continuation of the rectilinear path of the incident ray. If the angles at A, B and C are denoted by α, β and γ, respectively, then

$$\alpha+2\beta+\gamma=360°; \qquad (1)$$

and if the angles of incidence and refraction at the face BC are denoted by θ, θ', then

$$\theta=\frac{\gamma}{2}, \qquad \theta'=\frac{\alpha}{2}-\beta; \qquad (2)$$

and, finally, if the index of refraction is denoted by n,

$$n.\sin\theta'=\sin\theta. \qquad (3)$$

Consequently, eliminating the angles θ, θ' by means of (2) and (3), we obtain:

$$n.\sin(\frac{\alpha}{2}-\beta)=\sin\frac{\gamma}{2}; \qquad (4)$$

so that if the value of one of the angles α, β and γ is chosen arbitrarily, the other two angles can be determined by means of equations (1) and (4).

If the principal section of a KESSLER prism has the form of a rhombus (Fig. 217), parallel incident rays may be re-

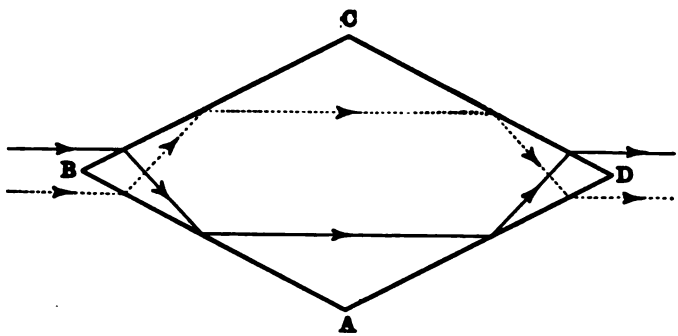


FIG. 217.—Rhomboidal form of KESSLER prism.

ceived on both faces BA and BC. In this case the angles α and γ are equal, and hence $\beta+\gamma=180°$, and therefore

$$\theta=\frac{\gamma}{2}, \qquad \theta'=\frac{3\gamma}{2}-180°,$$

so that

$$n.\sin\left(\frac{3\gamma}{2}-180°\right)=\sin\frac{\gamma}{2},$$

whence we obtain:

$$\sin\frac{\beta}{2}=\cos\frac{\gamma}{2}=\sqrt{\frac{n-1}{4n}}.$$

For example, if $n=1.64$, we find $\beta=36°\ 24', \gamma=143°\ 36'$.

173. Achromatic Combination of Two Thin Lenses.—The positions of the principal and focal points of a lens-system vary for light of different colors, and if the system is to be used as a magnifying glass or as the so-called ocular of a microscope or telescope, a chief consideration will be that the apparent sizes of the colored virtual images which are presented to the eye shall all be the same, that is, that the red and blue images, for example, shall subtend the same angle at the eye, no matter whether their actual sizes and positions are different or not. But the apparent size of the infinitely distant image of an object lying in the primary focal plane of the lens-system is measured by the

refracting power of the system (§ 122); and hence the condition of achromatism in this case is that the refracting powers (or focal lengths) of the system shall be equal for the two colors in question. (Achromatism with respect to the focal length; see § 167.)

Let us assume that the system is composed of two thin lenses whose refracting powers for light of a certain definite wave-length λ are denoted by F_1 and F_2; then the refracting power of the combination will be $F=F_1+F_2-c.F_1.F_2$, where c denotes the air-interval between the two lenses. For a second color of wave-length $\lambda+\Delta\lambda$ (where $\Delta\lambda$ denotes a small variation in the value of λ), the refracting powers of the lenses will be slightly different, and the refracting power of the combination for this color will be:

$$F+\Delta F=(F_1+\Delta F_1)+(F_2+\Delta F_2)-c(F_1+\Delta F_1)(F_2+\Delta F_2).$$

Subtracting these two equations, at the same time neglecting the term which involves the product of the small variations ΔF_1 and ΔF_2, we obtain:

$$\Delta F=\Delta F_1+\Delta F_2-(F_2.\Delta F_1+F_1.\Delta F_2)c.$$

Evidently, the condition that the system shall be achromatic with respect to its refracting power is $\Delta F=0$; which, therefore, is equivalent to the following:

$$c=\frac{F_2.\Delta F_1+F_1.\Delta F_2}{\Delta F_1+\Delta F_2}.$$

Now if n_1 denotes the index of refraction of the first lens for light of wave-length λ, then

$$F_1=(n_1-1)\mathrm{K}_1,$$

where K_1 denotes a constant whose value depends simply on the form of the infinitely thin lens, that is, on the curvatures of its surfaces. Similarly, for light of wave-length $\lambda+\Delta\lambda$, we have:

$$F_1+\Delta F_1=(n_1+\Delta n_1-1)\mathrm{K}_1;$$

and hence

$$\Delta F_1=\mathrm{K}_1.\Delta n_1=F_1\frac{\Delta n_1}{n_1-1}.$$

But $\Delta n_1/(n_1-1)=1/\nu_1$ is the expression for the dispersive power of the material of the first lens (§ 165), and accordingly we may write:

$$\Delta F_1=\frac{F_1}{\nu_1};$$

and, analogously, for the second lens:

$$\Delta F_2=\frac{F_2}{\nu_2}.$$

Introducing these expressions for ΔF_1 and ΔF_2 in the equation above, we find, therefore, as the condition that a pair of thin lenses shall be achromatic with respect to the refracting

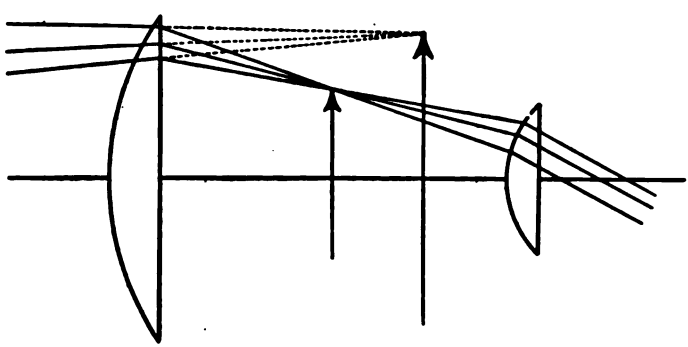


FIG. 218.—HUYGENS's ocular.

power of the system, the requirement that the distance between the two thin lenses shall satisfy the following equation:

$$c=\frac{\nu_2.F_1+\nu_1.F_2}{(\nu_1+\nu_2)F_1.F_2};$$

or

$$c=\frac{\nu_1.f_1+\nu_2.f_2}{\nu_1+\nu_2},$$

where $f_1=1/F_1$ and $f_2=1/F_2$ denote the focal lengths of the lenses.

If *both lenses are made of the same glass*, then $\nu_1=\nu_2$, so that in this case the condition of achromatism becomes:

$$c=\frac{f_1+f_2}{2}.$$

Thus, for example, HUYGENS's ocular (Fig. 218) is composed of two plano-convex lenses made of the same kind of glass, the curved face of each lens being turned away from the eye and towards the incident light. The first lens is called the "field-lens" and the second lens is called the "eye-lens." In this combination $f_1=2f_2$ (although in actual systems this

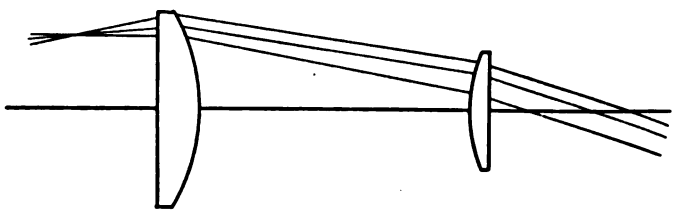


FIG. 219.—RAMSDEN'S ocular.

condition is usually only approximately satisfied) and $c=3f_2/2$, or $f_2:c:f_1=2:3:4$. RAMSDEN'S ocular (Fig. 219) consists likewise of two plano-convex lenses of the same kind of glass, but with their curved faces turned towards each other and in this combination $f_1=f_2=f=c$. Both of these types satisfy, therefore, the above condition of achromatism and yield images that are free from color-faults not only in the center but at the border of the field.

174. Achromatic Combination of Two Thin Lenses in Contact.—If the two lenses are in contact ($c=0$), the condition of achromatism, as found in the preceding section, becomes:

$$\nu_1 f_1+\nu_2 f_2=0,$$

or

$$\frac{F_1}{\nu_1}+\frac{F_2}{\nu_2}=0.$$

The quotient of the refracting power of a lens by the dis-

persive power of the glass of which it is made, namely, the magnitude F/ν, is sometimes called the *dispersive strength* of the lens; so that according to the above equation we may say that the condition of achromatism of a combination of two thin lenses in contact is that the algebraic sum of their dispersive strengths shall vanish. Accordingly, it appears that such a system can be achromatic only in case the substances of which the two lenses are made are different. Moreover, while one of the lenses must be convex and the other concave, their actual forms are of no consequence so far as the mere correction of the chromatic aberration is concerned. It is to be remarked also that in an achromatic lens of negligible thickness achromatism with respect to the focal lengths implies also achromatism with respect to the positions of the focal points and principal points, so that such a lens will be achromatic for all distances of the object.

If F denotes the prescribed refracting power of the combination then, since,

$$F = F_1 + F_2,$$

we find:

$$F_1 = \frac{\nu_1}{\nu_1 - \nu_2} F, \qquad F_2 = -\frac{\nu_2}{\nu_1 - \nu_2} F.$$

The total refracting power F will have the same sign as that of the lens which has the greater ν-value; for example, the combination will act like a convex lens provided the ν-value of the positive element exceeds that of the negative element.

Thus, being given the values of F, ν_1 and ν_2, we can employ the above relations to determine the required values of F_1 and F_2. Moreover, if K_1 denotes the algebraic difference of the curvatures of the two faces of the first lens, and, similarly, if K_2 denotes the corresponding magnitude for the second lens, then

$$K_1 = \frac{F_1}{n_1 - 1}, \qquad K_2 = \frac{F_2}{n_2 - 1},$$

where n_1, n_2 denote the indices of refraction of the two kinds

of glass for some standard wave-length, as already stated, which is usually light corresponding to the FRAUNHOFER D-line. Thus, while the magnitudes denoted by K_1 and K_2 may be computed, the actual curvatures or radii of the lens-surfaces remain indeterminate; so that there are still two

FIG. 220.—DOLLOND'S telescope objective.

FIG. 221. — FRAUNHOFER'S telescope objective, No. 1.

FIG. 222. — FRAUNHOFER'S telescope objective, No. 2.

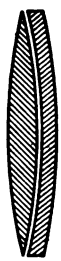


FIG. 223.—HERSCHEL'S telescope objective.

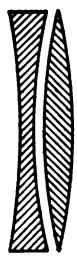


FIG. 224.—BARLOW'S telescope objective.

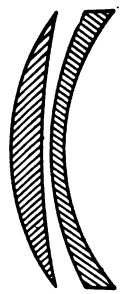


FIG. 225. — GAUSS'S telescope objective.

other conditions which may be imposed on an achromatic combination of this kind. For example, in some cases it may be conventient to cement the two components together, and then one of the conditions will be that the curvatures of the two surfaces in contact shall be equal. Usually, however, a more important requirement will be the abolition of two of the so-called spherical errors due to the fact that the

rays are not paraxial, so that the image will be sharp and distinct, especially at the center.

Some historic types of achromatic object-glasses of a telescope are illustrated in the accompanying diagrams. DOLLOND's achromatic doublet (Fig. 220) consisted of a double convex crown glass lens combined with a double concave flint glass lens; whereas FRAUNHOFER's constructions show a combination of a double convex and a plano-concave lens (Fig. 221) and of a double convex and a meniscus lens (Fig. 222). J. HERSCHEL's form (1821) is shown in Fig. 223, BARLOW's (1827) in Fig. 224; and, finally, the GAUSS type made by STEINHEIL in 1860 is exhibited in Fig. 225. The newer varieties of Jena glass make it possible to construct an achromatic objective of two lenses which is far superior in achromatism to any of the older types above mentioned.

PROBLEMS

1. Find the values of the reciprocals of the dispersive powers (§ 165) of alcohol and water, using data given in table in § 163. Ans. Alcohol, 60.5; water, 55.7.

2. The indices of refraction of rock salt for the FRAUNHOFER lines C, D and F are 1.5404, 1.5441 and 1.5531, respectively. Calculate the value of the reciprocal of the dispersive power. Ans. 42.84.

3. White light is emitted from a luminous point on the axis of a thin lens. If the yellow rays are brought to a focus at a point whose distance from the lens is denoted by u', show that the distance between the foci of the red and blue rays is approximately equal to $2\,F.u'/\nu$, where F denotes the refracting power of the lens for yellow light and ν denotes the reciprocal of the dispersive power of the lens-medium.

4. A lens is made of borate flint glass for which $\nu = 55.2$. The focal length of the lens for sodium light is 30 inches. Find the distance between the red and blue images of the sun formed by the lens. Ans. 0.54 in.

5. A crown glass prism of refracting angle 20° is to be combined with a flint glass prism so that the combination will be achromatic for the FRAUNHOFER lines C and F. The indices of refraction are as follows:

	n_C	n_D	n_F
Crown	1.526 849	1.529 587	1.536 052
Flint	1.629 681	1.635 036	1.648 260

Using the approximate formulæ for thin prisms, show that the refracting angle of the flint prism will be 9° 54′ 11″, and that the deviation of the rays corresponding to the D-line will be 4° 18′ 7″.

6. A direct vision prism combination is to be made with the same kinds of glass as in the preceding problem; so that rays corresponding to the D-line are to emerge without deviation. If the refracting angle of the crown glass prism is 20°, show that the refracting angle of the flint glass prism will be 16° 40′ 48″, and that the angular dispersion between C and F will be 9′ 33″.

7. An AMICI direct vision prism (§ 171) is to be made of crown glass and flint glass whose indices of refraction for the D-line are 1.5159 and 1.9626, respectively. If the refracting angles of the two equal crown glass prisms are each equal to 45°, show that the refracting angle of the middle flint glass prism will be 98° 7.4′.

8. A KESSLER prism (§ 172) in the form of a rhombus is made of glass of index $n_D = 1.6138$. Find the angles of the prism. Ans. 35° 5′ and 144° 55′.

9. A thin lens is made of crown glass for which $\nu_1 = 60.2$. Another thin lens is made of flint glass for which $\nu_2 = 36.2$. When the two lenses are placed in contact they form an achromatic combination of focal length 10 cm. Find the focal length of each lens. Ans. $f_1 = 3.99$ cm.; $f_2 = -6.63$ cm.

10. An achromatic doublet is to be made of two thin lenses cemented together, and the focal length of the combination for the D-line is to be 25 cm. The first lens is a symmetric convex lens of barium silicate glass and the other

lens is a concave lens of sodium lead glass. The indices of refraction are:

	n_D	$n_F - n_C$
Barium silicate	1.6112	0.01747
Sodium lead	1.5205	0.01956

Find the radii of the surfaces on the supposition that the rays corresponding to the lines C and F are united.

Ans. The radii of the first and last surfaces are $+14.60$ and -22.65 cm., respectively.

11. A symmetric double convex lens is made of rock salt for which $n_C = 1.5404$ and $n_F = 1.5531$. Find the thickness of the lens if the focal lengths for the colors C and F are equal.

Ans. $d = 3.4363.r$, where r denotes the radius of the first surface of the lens.

12. Two thin lenses of the same kind of glass, one convex of focal length 9 inches, the other concave of focal length 4 inches, are separated by an interval of 20 inches. A small white object is placed 36 inches in front of the convex lens. Show that the various colored images are all formed at the same place.

13. Two thin lenses of the same kind of glass, one convex and the other concave, and both of focal length 4 inches, are adjusted on the same axis until the colored images of a white object placed 12 inches in front of the convex lens are formed at the same place. Show that the interval between the lenses must be twelve inches.

14. A lens-system surrounded by air is composed of m spherical refracting surfaces. Assuming that the total thickness of the system is negligible, show that the condition of achromatism is

$$\sum_{k=2}^{k=m} (R_{k-1} - R_k)\, \delta n_k = 0,$$

where R_k denotes the curvature of the kth surface and δn_k denotes the dispersion of the medium included between the $(k-1)$th and kth surfaces for light of the two colors to be compensated.

CHAPTER XV

RAYS OF FINITE SLOPE. SPHERICAL ABERRATION, ASTIGMATISM OF OBLIQUE BUNDLES, ETC.

175. Introduction.—The theory of the symmetrical optical instrument, as it has been developed in the preceding chapters, is based on the assumption that the rays concerned in the formation of the image are entirely confined to the so-called paraxial rays (§ 63) whose paths throughout the system are contained within an exceedingly narrow cylindrical region of space immediately surrounding the axis. With this fundamental restriction it was found that there was perfect collinear correspondence between object-space and image-space; so that a train of spherical waves emanating from an object-point was transformed by the optical system into another train of spherical waves accurately converging to or diverging from a corresponding center called the image-point; and so that, in general, a plane object at right angles to the axis was reproduced point by point by a similar plane image. As a matter of fact, these ideal conditions are never realized in any actual optical system except in the case of a plane mirror or combination of plane mirrors. Moreover, according to the wave-theory of light, a mere homocentric convergence of the rays is not sufficient for obtaining a point-image of a point-source; for this theory lays particular stress on the further essential requirement that the effective portion of the wave-surface which contributes to the production of the image shall be relatively large in comparison with the radius of the surface, if the light-effect is to be concentrated as nearly as possible at a single point and not spread over some considerable area in the vicinity of the point. This condition implies, therefore, that the aperture of the bundle

of effective rays must not be below a certain finite limit, in other words we are compelled by a practical necessity, wholly aside from the principles at the basis of geometrical optics, to employ more or less wide-angle bundles of rays. Moreover, if a wide-angle bundle of rays is a requirement of a distinct, clear-cut image, it is also equally essential for a bright image. Thus, on both theoretical and practical grounds, it is found necessary to extend the limits of the effective rays beyond the paraxial region.

Instead, therefore, of the ideal case of collinear correspondence of object-space and image-space, the theory of optical instruments is complicated by numerous practical and, for the most part irreconcilable difficulties, due chiefly to the so-called *aberrations* or failure of the rays to arrive at the places where they might be expected according to the simple theory of collineation or point-to-point correspondence (punctual imagery). In the preceding chapter brief reference was made to the *chromatic aberrations* arising from the differences in the color of the light; but now we have to deal with the *monochromatic aberrations* of rays of light of one definite wave-length which are caused by the peculiarities of the curved surfaces at which the rays are reflected and refracted. These surfaces are nearly always spherical in form, and hence the aberrations of this latter kind are usually called *spherical aberrations*. A complete treatment of this intricate subject lies wholly outside the scope of this volume. In the present chapter it must suffice to point out the general nature of some of the more important of the so-called spherical errors. First, however, we must see how to trace the path of a single ray through a centered system of spherical surfaces before we are in a position to study a bundle of rays.

176. Construction of a Ray Refracted at a Spherical Surface.—In § 34 a method was explained for constructing the path of a ray refracted from one medium into another, which is always applicable to a refracting surface of any form.

The following elegant and useful construction of the path of a ray refracted at a spherical surface was published in 1807 by THOMAS YOUNG (1773–1829.)

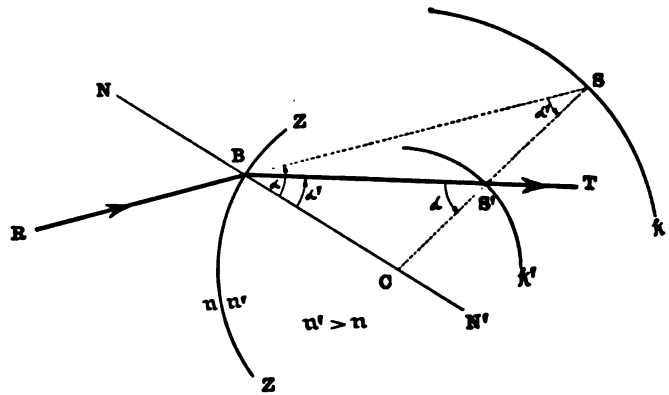


FIG. 226.—Construction of ray refracted at convex spherical surface ($n' > n$).

In the accompanying diagrams (Figs. 226 to 229) the center of the spherical refracting surface ZZ is designated by C. The point R is any point on the path of the incident ray lying in the first medium of refractive index n. The point

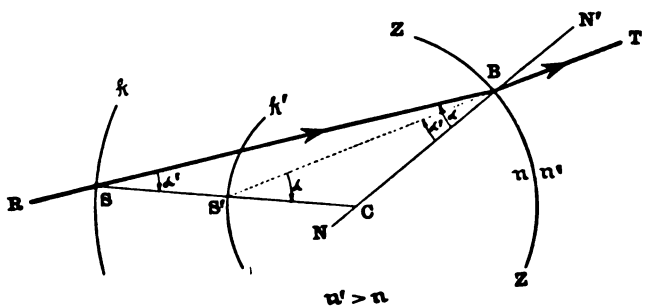


FIG. 227.—Construction of ray refracted at concave spherical surface ($n' > n$).

where the ray meets the spherical refracting surface is marked B. The plane of the paper which contains the incident ray RB and the incidence-normal BC is the plane of incidence.

The index of refraction of the second medium is denoted by n' and the radius of the spherical refracting surface by r. Around C as center and with radii equal to $n'.r/n$ and $n.r/n'$

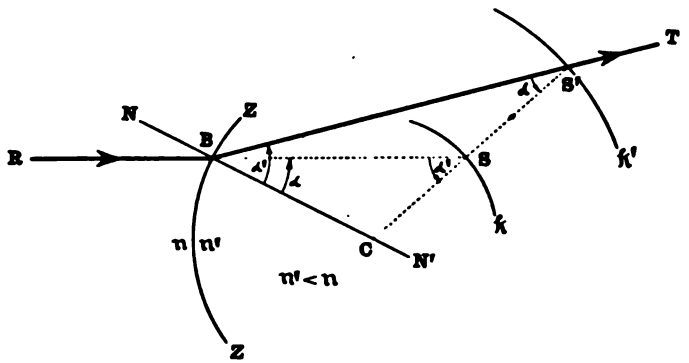


FIG. 228.—Construction of ray refracted at convex spherical surface ($n'<n$).

describe, in the plane of incidence, the circular arcs k and k', respectively; and let S designate the point where the straight line RB, produced if necessary, meets the arc k. Draw the straight line CS ntersecting the arc k' in the point S'. Then

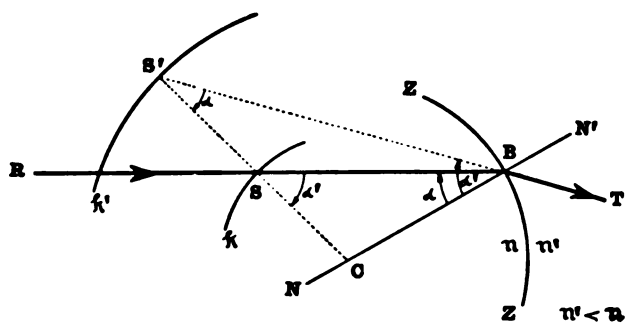


FIG. 229.—Construction of ray refracted at concave spherical surface ($n'<n$).

the straight line BT drawn from B through S' will represent the path of the refracted ray. In making this construction, care must be taken to select for the point S that one of the

two points in which the straight line RB cuts the circle k which will make the segments BS and BS′ both fall on the same side of the incidence-normal, since the angles of incidence and refraction are described always in the same sense, both clockwise or both counter-clockwise.

The proof of the construction is simple. Since the radius $r=BC$ is a mean proportional between the radii $SC=n'.r/n$ and $S'C=n.r/n'$, that is, since

$$CS : CB = CB : CS' = n' : n,$$

the triangles CBS and CBS′ are similar, and hence $\angle CBS = \angle BS'C$. In the triangle CBS:

$$\sin\angle CBS : \sin\angle BSC = CS : CB = n' : n.$$

By the law of refraction: $n.\sin\alpha = n'.\sin\alpha'$, where $\alpha = \angle CBS$. Consequently, $\angle BSC = \angle CBS' = \alpha'$, so that the straight line BS′ is the path of the refracted ray.

This construction can be employed to trace the path of a ray graphically from one surface to the next through a centered system of spherical refracting surfaces.

177. The Aplanatic Points of a Spherical Refracting Surface.—Incidentally, in connection with the preceding construction, attention is directed to the singular character of all pairs of points such as S, S′ determined by the intersections of the two concentric auxiliary spherical surfaces with any straight line drawn from their common center C. To every incident ray directed towards the point S there will correspond a refracted ray which will pass ("really" or "virtually") through the other point S′; so that in this special case we obtain a homocentric bundle of refracted rays from a homocentric bundle of incident rays, for all values of the aperture-angle of the bundle. Thus, S′ is a point-image of the object-point S. The distances of S and S′ from the center C are connected by the invariant-relation:

$$n'.CS' = n.CS.$$

That pair of these points which lies on the optical axis is especially distinguished and called the pair of *aplanatic*

points of the spherical refracting surface; they are designated by J, J′ (Fig. 230). Thus, we have:

$$CJ : AC = AC : CJ' = n' : n,$$

or

$$CJ.CJ' = r^2, \quad n.CJ = n'.CJ'.$$

The aplanatic points, therefore, lie always on the same side

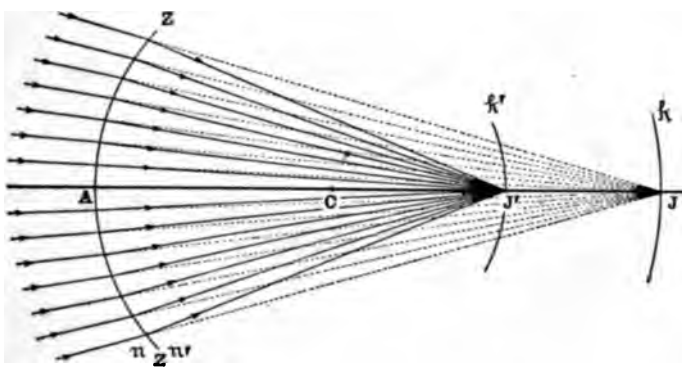


FIG. 230.—Aplanatic points of spherical refracting surface.

of the center C so that whereas the rays must pass "really" through one of them, they will pass "virtually" through the other. In geometrical language the points J, J′ are said to be harmonically separated (§ 67) by the extremities of the axial diameter of the refracting sphere.

178. Spherical Aberration Along the Axis.—However, in general, a homocentric bundle of rays incident on a spherical refracting surface will not be homocentric after refraction. The diagram (Fig. 231) represents the case of a meridian section of a bundle of incident rays which are all parallel to the axis of a convex spherical refracting surface for which $n' > n$. It will be seen that, whereas the paraxial rays after refraction meet on the axis at the second focal point F′, the outermost or edge rays cross the axis at a point L′ between the vertex A and the focal point F′; and the intermediate rays cross the axis at points lying between F′ and L′. The segment F′L′ is the measure of the *spherical aberration along*

the axis or the axial aberration of the edge ray of a direct cylindrical bundle of incident rays. (By a "direct" bundle of rays is meant a bundle of rays emanating from a point on the axis.) In the figure this segment is negative, that is, measured in the sense opposite to that of the incident light; and

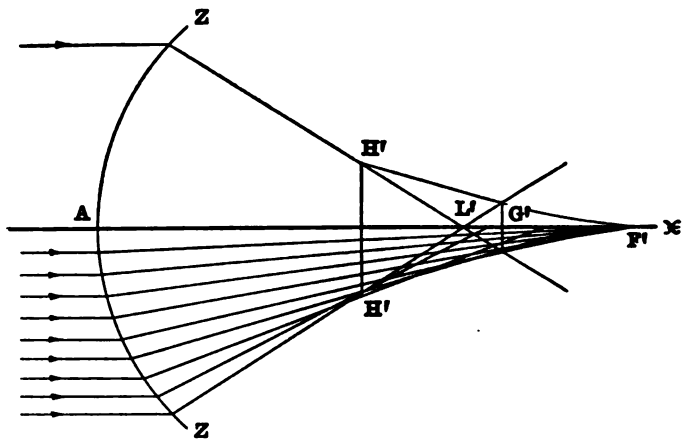


FIG. 231.—Spherical aberration.

this effect is usually described by saying that a convex spherical refracting surface at which light is refracted from air to glass is *spherically under-corrected;* whereas, under the same circumstances, a concave spherical refracting surface will be found to be spherically *over-corrected*, that is, the segment F'L' in this case will be positive. In fact, the points of intersection of pairs of consecutive rays lying in the plane of a meridian section of a spherical refracting surface form a curved line lying symmetrically above and below the axis, if the bundle of incident rays is symmetric with respect to the axis; and this plane curve is the so-called *caustic curve* of the meridian rays. The two branches on opposite sides of the axis unite in a double point or cusp at the point on the axis where the paraxial rays intersect, so that the axis is tangent to both branches at this point, which in the figure

is the point F'. The system is said to be spherically over-corrected or under-corrected according as the cusp is turned towards the incident light (<) or away from it (>), respectively; on the supposition that the incident rays are parallel to the axis. Each refracted ray in the meridian plane touches the caustic curve, and hence this curve is said to be the geometrical envelope of the meridian section of the bundle of refracted rays.

If the entire figure is revolved around the optical axis the arc ZZ will generate a zone of the spherical refracting surface containing the vertex A; and each incident ray proceeding parallel to the axis will generate a cylindrical surface, and all the refracted rays corresponding to the incident rays which lie on the surface of one of these cylinders will intersect in one point lying on the axis between F' and L'. The revolution of the caustic curve will generate a *caustic surface*, which will be the enveloping surface of the bundle of refracted rays (see § 187.)

The caustic curve terminates at the point H' where the edge ray intersects the next consecutive ray in the meridian section. If a plane screen erected at right angles to the axis so as to catch the light transmitted by the bundle of refracted rays is placed initially in the transversal plane that passes through the extreme point H' and then gradually shifted parallel to the axis towards the second focal plane, there will appear on the screen at first a circular patch of light surrounded on its outer edge by a brighter ring, which will gradually contract as the screen approaches L'. Between L' and F' there will be seen at the center of the circular patch of light an increasingly bright spot. For a certain position G' where the distance of the screen from F' is about three-fourths of the length of F'L' the cross-section of the bundle of refracted rays will have its narrowest contraction. This section is sometimes called the *least circle of aberration*.

179. Spherical Zones.—Since, in general, it is not possible to abolish the spherical aberration of a single spherical re-

fracting surface, the only means available is to try to accomplish this result by distributing the duty of refracting the rays over a series of surfaces whose curvatures and distances apart are so nicely adjusted with respect to each other that when the rays finally emerge they will all unite in one focus on the axis. Thus, for example, if the incident rays are supposed to be parallel to the axis of the system, and if the system has been designed so as to be spherically corrected

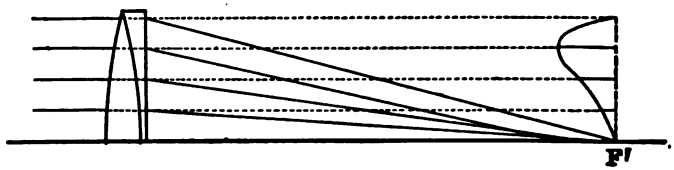


FIG. 232.—Graphical representation of the spherical zones of a lens.

for the edge ray which meets the first surface at the distance h from the axis, it is conceivable that all the intermediate rays of incidence-heights z (where $h > z > 0$) might perchance emerge from the system along paths which all likewise passed through the focal point F′; but practically this never happens. If the edge ray intersects the axis at F′, an intermediate ray of incidence-height z will cross the axis at some other point L′, and the segment F′L′ is called the *spherical aberration of the zone of radius z* or simply the *spherical zone z*. The spherical zones of a lens may be exhibited graphically by plotting a curve whose abscissæ are the values of F′L′ and whose ordinates are the corresponding values of z, as represented in Fig. 232.

180. Trigonometrical Calculation of a Ray Refracted at a Spherical Surface.— The diagram (Fig. 233) represents a meridian section ZZ of a spherical refracting surface of radius r (=AC) separating two media of indices of refraction n, n'. A ray RB incident on the surface at B at an angle $\alpha = \angle NBR = \angle CBL$ crosses the axis at L at a slope-angle $\theta = \angle ALB$. If the central angle is denoted by $\phi = \angle BCA$,

and if the abscissa of the point L with respect to the center C is denoted by c, that is, if $c = \mathrm{CL}$, then in the triangle CBL, we have the relations:

$$\alpha = \theta + \phi, \qquad c.\sin\theta = -r.\sin\alpha.$$

The path of the corresponding refracted ray is shown by the straight line BT which crosses the axis at the point L′; and

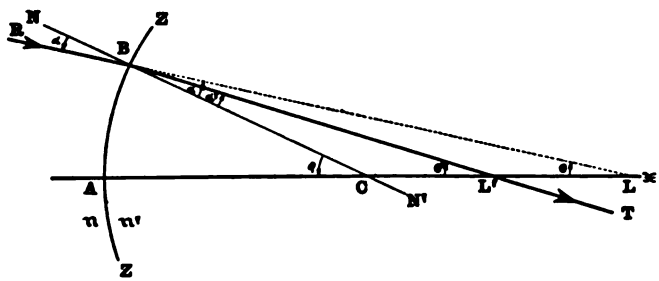


Fig. 233.—Diagram for trigonometrical calculation of refracted ray.

if we put $\alpha' = \angle \mathrm{N'BL'}$, $\theta' = \angle \mathrm{AL'B}$ and $c' = \mathrm{CL'}$, we obtain a similar pair of formulæ from the triangle CBL′, namely:

$$\alpha' = \theta' + \phi, \qquad c'.\sin\theta' = -r.\sin\alpha'.$$

Accordingly, being given the constants denoted by n, n' and r, and the parameters $(c,\ \theta)$ of the incident ray, we can find the parameters $(c',\ \theta')$ of the refracted ray by means of the following system of equations:

$$\sin\alpha = -\frac{c}{r}\sin\theta,$$

$$\sin\alpha' = \frac{n}{n'}\sin\alpha, \qquad \theta' = \theta + \alpha' - \alpha,$$

$$c' = -r\frac{\sin\alpha'}{\sin\theta'}.$$

It is easy to see that if we have given two incident rays which both cross the axis at the same point L, so that the abscissa c has the same value for both rays while the slope-angles θ are different, different values of c' will, in general, be obtained for the abscissæ of the points where the two cor-

responding refracted rays cross the axis. This is the analytical statement of the fact of spherical aberration (§ 178).

The formulæ for calculating the path of a ray reflected at a *spherical mirror* may be derived immediately by putting $n' = -n$ (§ 75) in the preceding system of equations. Thus we find:

$$\sin \alpha = -\frac{c}{r}\sin\theta, \quad \alpha' = -\alpha, \quad \theta' = \theta - 2\alpha, \quad c' = r\frac{\sin\alpha}{\sin(\theta - 2\alpha)}.$$

Incidentally, a number of other useful relations may be obtained from Fig. 233. For example, if the distances of the points L and L′ where the ray crosses the axis before and after refraction measured from the incidence-point B are denoted by l and l', respectively, that is, if $l = \mathrm{BL}$, $l' = \mathrm{BL}'$, where l and l' are to be reckoned positive or negative according as these lengths are measured in the same direction as the light traverses the ray or in the opposite direction, respectively; then

$$l'.\sin\theta' = l.\sin\theta;$$

and, since by the law of refraction,

$$n'.c'.\sin\theta' = n.c.\sin\theta,$$

we obtain the useful invariant relation:

$$\frac{n'.c'}{l'} = \frac{n.c}{l}.$$

Moreover, by projecting the two sides c and l of the triangle CBL on the third side r, the following formula is obtained:

$$r = l.\cos\alpha - c.\cos\phi,$$

which may be written:

$$\frac{c}{l} = \frac{r}{\cos\phi}\left(\frac{\cos\alpha}{r} - \frac{1}{l}\right).$$

Similarly, in the triangle CBL′:

$$\frac{c'}{l'} = \frac{r}{\cos\phi}\left(\frac{\cos\alpha'}{r} - \frac{1}{l'}\right).$$

Multiplying the first of these equations by n and the second by n' and equating the resulting expressions, we find:

$$n'\left(\frac{\cos\alpha'}{r} - \frac{1}{l'}\right) = n\left(\frac{\cos\alpha}{r} - \frac{1}{l}\right),$$

which may also be written:

$$\frac{n'}{l'}-\frac{n}{l}=\frac{n'.\cos a'-n.\cos a}{r}=D \text{ (say)};$$

or finally:

$$L'=L+D,$$

where $L=n/l$, $L'=n'/l'$.

If the ray is a *paraxial ray*, we may put $\cos a=\cos a'=1$ (§ 63); and now if we write u, u' in place of l, l', respectively, the formula above will reduce to the abscissa-equation for the refraction of a paraxial ray at a spherical surface (§ 78).

Moreover, if in the last formula we put $n'=n$ (§ 75), we find the corresponding relation for the *reflection of a ray at a spherical mirror*, namely:

$$\frac{1}{l}+\frac{1}{l'}=\frac{2\cos a}{r}.$$

181. Path of Ray through a Centered System of Spherical Refracting Surfaces. Numerical Calculation.—Using the same system of notation as in § 118, we may write the formula for the refraction of a paraxial ray at the kth surface of a centered system of spherical refracting surfaces, as follows:

$$U_k'=U_k+F_k,$$

where

$$U_k=n_k/u_k, \quad U_k'=n_{k+1}/u_k', \quad \text{and } F_k=(n_{k+1}-n_k)/r_k;$$
$$u_k=A_kM_k, \quad u_k'=A_kM_{k+1}, \quad r_k=A_kC_k.$$

And if $d_k=A_kA_{k+1}$, then also:

$$1/U_{k+1}=1/U'_k-d_k/n_{k+1}.$$

According to the relations given in § 180, we have the following system of formulæ for the refraction at the kth surface of a ray whose slope-angles before and after refraction have the finite values $\theta_k=\angle A_kL_kB_k$ and $\theta_{k+1}=\angle A_kL_{k+1}B_k$, respectively:

$$\sin a_k=-\frac{c_k}{r_k}.\sin\theta_k, \qquad \sin a_k'=\frac{n_k}{n_{k+1}}.\sin a_k,$$

$$\theta_{k+1}=\theta_k+\theta_k'-a_k, \qquad c_k'=-r_k.\frac{\sin a_k'}{\sin\theta_{k+1}},$$

where $c_k = C_k L_k$ and $c_k' = C_k L_{k+1}$. Moreover, if we put

$$a_k = C_k C_{k+1} = d_k + r_{k+1} - r_k,$$

then

$$c_{k+1} = c_k' - a_k.$$

In order to exhibit the methods of calculations by means of these formulæ, a comparatively simple numerical illustration is appended. The actual example here chosen is one given by Dr. MAX LANGE in his paper entitled "Vereinfachte Formeln für die trigonometrische Durchrechnung optischer Systeme" (Leipzig, 1909), pages 24, foll. The optical system is a two-lens object-glass of a telescope for which the data were published by Dr. R. STEINHEIL in the *Zeitschrift für Instrumentenkunde*, xvii (1897), p. 389, as follows:

Indices of refraction (for D-line):

$n_1 = n_3 = n_5 = 1$ (air); $n_2 = 1.614\ 400$ (flint); $n_4 = 1.518\ 564$ (crown).

Thicknesses:

$$d_1 = 2;\ d_2 = 0.01;\ d_3 = 5.$$

Radii:

$r_1 = +420$; $r_2 = +181.995$; $r_3 = +178.710$; $r_4 = -40\ 133.8$.

The incident rays are parallel to the axis, so that

$$\theta_1 = 0, \qquad u_1 = c_1 = \infty \ (U_1 = 0).$$

The calculation is divided into two parts, namely: (1) the calculation of the paraxial ray, and (2) the trigonometric calculation of the edge ray which meets the first surface of the object-glass at the height $h_1 = 33$ above the axis. When $c_1 = \infty$, we find $\sin \alpha_1 = h_1/r_1$, which, according to the above data, gives $\lg \sin \alpha_1 = 1.5185139$. This is the starting point of the calculation of the edge ray.

Each vertical column contains the calculation for one spherical refracting surface. The sign written after a logarithm indicates the sign of the number to which the logarithm belongs. Generally the calculations do not have to be performed to the degree of accuracy to which they are carried here.

1. PARAXIAL RAY

	$k=1$	$k=2$	$k=3$	$k=4$
$\lg(n_{k+1}-n_k)$	9.7884512+	9.7884512−	9.7148024+	9.7148024−
clg r_k	7.3767507+	7.7399406+	7.7478511+	5.3964898−
lg F_k	7.1652019+	7.5283918−	7.4626535+	5.1112922+
F_k	+0.00146286	−0.00337592	+0.00290171	+0.00001292
U_k	0.00000000	+0.00146552	−0.00191036	+0.00099459
U_k'	+0.00146286	−0.00191040	+0.00099135	+0.00100751
clg U_k'	2.8347972+	2.7188755−	3.0037761+	2.9967506+
lg d_k	0.3010300+	8.0000000+	0.6989700+	
clg n_{k+1}	9.7919889+	0.0000000+	9.8185669+	
$\lg(d_k/n_{k+1})$	0.0930189+	8.0000000+	0.5175369+	
$-d_k/n_{k+1}$	− 1.2389	− 0.0100	− 3.2926	
$1/U_k'$	+683.5924	−523.4504	+1008.7327	
$1/U_{k+1}$	+682.3535	−523.4604	+1005.4401	
lg U_{k+1}	7.1659906+	7.2811162−	6.9976438+	

$$\lg u_4' = \operatorname{clg} U_4' = 2.9967506+;\quad u_4' = A_4F' = +992.546$$

$$\operatorname{clg}(U_1'.U_2'.U_3'.U_4') = 1.5541994-$$
$$\lg(U_2.U_3.U_4) = 1.4447506-$$
$$\lg f = 2.9989500+$$
$$f = +997.585$$

2. EDGE RAY

	$k=1$	$k=2$	$k=3$	$k=4$
$-a_{k-1}$		+236.0050	+ 3.2750	+40307.51
$c_k'-_1$		+682.2850	−685.6727	+ 1353.49
c_k		+918.2900	−682.3977	+41661.00
lg c_k		2.9629799+	2.8340376−	4.6197297+
lg $\sin\theta_k$		8.4765370−	8.8114112+	8.3325613−
clg r_k		7.7399405+	7.7478511+	5.3964898−
lg $\sin a_k$	8.8952646+	9.1794574+	9.3932999+	8.3487808−
lg n_k/n_{k+1}	9.7919889+	0.2080111+	9.8185669+	0.1814331+
lg $\sin a_k'$	8.6872535+	9.3874685+	9.2118668+	8.5302139−
lg r_k	2.6232493+	2.2600595+	2.2521489+	4.6035102−
clg $\sin\theta_{k+1}$	1.5234630−	1.1885888+	1.6674387−	1.4803948−
lg c_k'	2.8339658+	2.8361168−	3.1314544+	4.6141189+

$$
\begin{array}{rrrr}
 & -a_1 = & -\ 4^\circ\ 30'\ 23.24'' & c_4' = +41126.23 \\
 & a_1' = & +\ 2^\circ\ 47'\ 22.69'' & r_4 = -40133.80 \\
\theta_1 = 0,\ a_1' - a_1 = & \theta_2 = & -\ 1^\circ\ 43'\ 0.55'' & A_4L_5 = +\ 992.43 \\
 & -a_2 = & -\ 8^\circ\ 41'\ 40.45'' & -u_4' = -\ 992.55 \\
 & & -10^\circ\ 24'\ 41.00'' & F'L_5 = -\ 0.12 \\
 & a_2' = & +14^\circ\ 7'\ 31.28'' & \\
 & \theta_3 = & +\ 3^\circ\ 42'\ 50.28'' & \\
 & -a_3 = & -14^\circ\ 19'\ 13.26'' & \\
 & & -10^\circ\ 36'\ 22.98'' & \\
 & a_3' = & +\ 9^\circ\ 22'\ 26.69'' & \\
 & \theta_4 = & -\ 1^\circ\ 13'\ 56.29'' & \\
 & -a_4 = & +\ 1^\circ\ 16'\ 45.13'' & \\
 & & +\ 0^\circ\ 2'\ 48.84'' & \\
 & a_4' = & -\ 1^\circ\ 56'\ 33.95'' & \\
 & \theta_5 = & -\ 1^\circ\ 53'\ 45.11'' &
\end{array}
$$

Thus, we see that this object-glass has a slight spherical aberration of -0.12, that is, it is a little under-corrected (§ 178).

182. The Sine-Condition or Condition of Aplanatism.— Suppose that for a certain object-point M (Fig. 234) on the axis of a symmetrical optical instrument the spherical aberration has been abolished for all the zones of the system, so that rays proceeding from this point will all be accurately focused at the conjugate image-point M′. On a straight line perpendicular to the axis at M take a point Q very close to M; and let $y' = M'Q'$ denote the size of the image of the object $y = MQ$ which is produced by the central zone, that is, by the paraxial rays. Now even though the system is spherically corrected with respect to the pair of axial points M, M′, it by no means follows that rays emanating from Q will all meet again in Q′. In order that this shall be the case, the magnification-ratio must be equal to y'/y for all the zones of the system. Draw the object-ray MB_1 and the corresponding image-ray B_2M'; if the slopes of these rays are

denoted by θ and θ', it may be shown that for the zone corresponding to the incidence-point B_1 the magnification-ratio is equal to $n.\sin\theta/n'.\sin\theta'$; and if this is equal to y'/y, then the image formed by rays belonging to this zone will be of the same size as the image y' made by the paraxial rays.

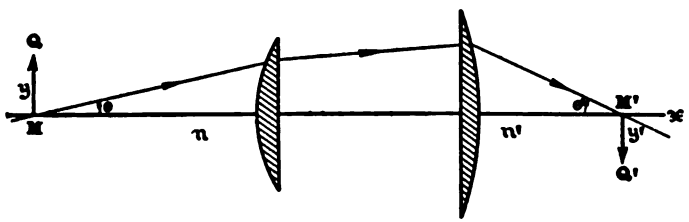


Fig. 234.—Sine-condition.

Thus, in order that with the employment of wide-angle bundles of rays a symmetrical optical instrument may produce a sharp image of a little plane element perpendicular to the axis of the instrument, not only must the system be spherically corrected for the pair of conjugate axial points M, M′, but it must also satisfy the so-called *Sine-Condition*, namely,

$$\frac{n.\sin\theta}{n'.\sin\theta'}=\frac{y'}{y}=\mathfrak{y}.$$

This celebrated principle was clearly formulated by Abbe in 1873, but it had already been recognized by Seidel, and it may be deduced from a general law of radiant energy which was first given by Clausius (1864). The proof of it must be omitted here. It may be stated in words as follows: The necessary and sufficient condition that all the zones of a spherically corrected system shall produce images of equal size at the point M′ conjugate to the axial point M is that, for all rays proceeding from M, the ratio of the sines of the slope-angles θ, θ' of each pair of corresponding incident and emergent rays shall be constant; that is,

$$\frac{\sin\theta}{\sin\theta'}=\frac{n'}{n}.\mathfrak{y}=\text{constant}.$$

The sine-condition

$$n.y.\sin\theta = n'.y'.\sin\theta'$$

is essentially different from the SMITH-HELMHOLTZ law for paraxial rays (see § 88 and § 118), namely, $n.y.\tan\theta = n'.y'.\tan\theta'$, although when the angles θ, θ' are small, both conditions may be expressed by the equation $n.y.\theta = n'.y'.\theta'$.

If the optical system is spherically corrected for the pair of axial points M, M', and if at the same time the sine-condition is satisfied, the points M, M' are called the *aplanatic pair of points* of the system. It may be demonstrated that no optical system can have more than one pair of such aplanatic points. In the case of a single spherical refracting surface the two points J, J' (§ 177) whose distances from the center C are such that

$$\mathrm{CJ}.\mathrm{CJ}' = r^2, \qquad n.\mathrm{CJ} = n'.\mathrm{CJ}',$$

are a pair of aplanatic points as above defined; for they are free from spherical aberration and if they are joined by straight lines BJ, BJ' with any point B on the spherical refracting surface, and if we put $\theta = \angle\mathrm{CJB}$, $\theta' = \angle\mathrm{CJ'B}$, we have $\sin\theta/\sin\theta' = n/n' = \text{constant}$. This property of the points J, J' of a refracting sphere has been ingeniously utilized in the construction of the objective of the compound microscope.

If in Fig. 234 we put $l = \mathrm{B_1M}$, then $\sin\theta = -h/l$, where h denotes the height of the point $\mathrm{B_1}$ above the axis. Hence, the sine-condition may be written:

$$\frac{h}{l.\sin\theta'} = \frac{n'}{n}\cdot y;$$

or since (§ 124)

$$y = \frac{f}{x} = -\frac{n}{n'}\cdot\frac{f'}{x},$$

where f, f' denote the focal lengths of the system and x denotes the abscissa of M with respect to the primary focal point F ($x = \mathrm{FM}$), we obtain also:

$$\frac{h}{\sin\theta'} = \frac{l.f'}{x}.$$

Suppose now that the object-point M is infinitely distant so that $x = l = \infty$; then for a ray parallel to the axis meeting the first surface at the height h, we shall have:

$$\frac{h}{\sin \theta'} = f'.$$

Thus, if the aplanatic points are the infinitely distant point of the axis and the second focal point F′, and if around F′ as center we describe a sphere of radius equal to f', the parallel object-rays will meet their corresponding image-rays on the surface of this sphere; whereas in the case of collinear imagery with paraxial rays the points of intersection of the incident and emergent rays under the same circumstances will all lie in the secondary principal plane (§ 119), which touches the sphere above mentioned at the point where the axis crosses it.

If therefore we put $h/\sin \theta' = e$, the sine-condition for an infinitely distant object is $e+f=0$. For example, in the case of the telescope objective calculated in § 181:

$$\begin{aligned} \lg h_1 &= 1.5185139 + \\ \operatorname{clg} \sin \theta_5 &= 1.4803948 - \\ \lg e &= 2.9989087 - \end{aligned} \qquad \begin{aligned} e &= -997.490 \\ f &= +997.585 \\ e+f &= +\ 0.095 \end{aligned}$$

Accordingly, the sine-condition is very nearly satisfied in the case of this object-glass.

183. Caustic Surfaces.—The characteristic geometrical property of a bundle of light-rays emanating originally from a point-source is expressed in a law announced by Malus in 1808 (§ 39), which may be stated in terms of the undulatory theory of light as follows: The rays of light are always normal to the wave-surfaces. In fact, what is meant by a wave-surface is any surface which cuts the rays orthogonally. In general, the curvatures of the normal sections at any point of a curved surface will vary from one azimuth to another; but, according to Euler's theorem (§ 111), the normal sections of greatest and least curvature, called the *principal sections* of

the surface at the point in question, are always at right angles to each other. It is well known that the normal drawn to any point of a curved surface will not meet the normal at a consecutive point taken arbitrarily. But if the consecutive point is taken in the direction of either of the principal sections, the two consecutive normals will intersect. Thus, along each normal to a curved surface there are two points called the principal centers of curvature (§ 111), where consecutive normals lying in the two principal sections intersect.

Accordingly, if we regard a bundle of rays of light as a system of normals to the wave-surface, we may say that each ray determines two principal sections of the bundle, and that, in general, there will be two points on the ray, the so-called *image-points* (*cf.* § 113), where contiguous rays in each of the two principal sections intersect the ray in question. The assemblage of these pairs of image-points on all the rays of a wide-angle bundle of rays emanating originally from a point-source form a surface of two sheets called the *caustic surface* (*cf.* § 42). Each ray of the bundle is tangent to both sheets of the caustic surface. In the special case when the bundle of rays is symmetrical about an axis, one sheet of the caustic surface will be a surface of revolution, whereas the other sheet will be a portion of the axis of symmetry (see § 178).

184. Meridian and Sagittal Sections of a Narrow Bundle of Rays before and after Refraction at a Spherical Surface.—The apertures of the bundles of effective rays which are transmitted through a symmetrical optical instrument are all limited by the position and dimensions of the aperture-stop (§ 134). For the present it will be assumed that the diameter of the stop is very small. Each point of the object lying in the field of view is the source of a narrow bundle of rays which contains one ray, called the *chief ray* (§ 140), which in traversing the medium where the stop is placed, passes through the center of the stop. Accordingly, the chief ray will lie in the meridian plane determined by the object-

point where the bundle of rays originates. The path of this chief ray may be traced geometrically by YOUNG's construction (§ 176) or it may be calculated trigonometrically by means of the system of formulæ given in § 181. We have now to investigate the positions on this chief ray of the two image-points produced by the intersections of this ray with the rays immediately adjacent to it lying in the two principal sections of the bundle as determined by its chief ray (§ 183). Whenever a narrow bundle of rays has two such image-points, it is said to be *a tigmatic*. Practically, this is always the case if the chief ray is incident on a refracting surface at an angle α which is not vanishingly small. Under such conditions the bundle of refracted rays will be astigmatic, and we have the case which some writers call "astigmatism by incidence" but which is better described as the *astigmatism of an oblique bundle of rays*, as distinguished from

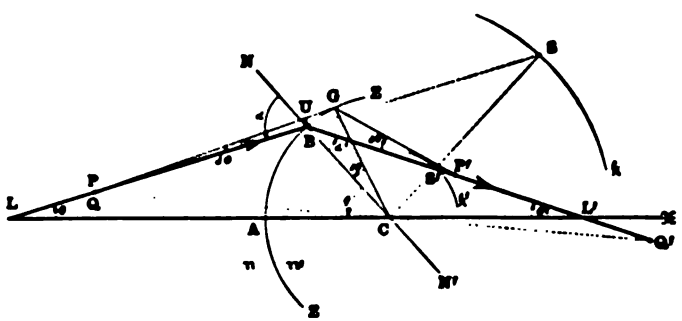


FIG. 235.—Meridian section of narrow bundle of rays refracted at spherical surface.

the astigmatism produced by direct (normal) incidence on an astigmatic refracting surface or surface of double curvature (Chapter IX).

In the diagrams (Figs. 235, 236), which show the meridian section ZZ of a spherical refracting surface whose center is at C and vertex at A (Fig. 235), the point designated by P (or Q) represents an object-point which is the source of a narrow

homocentric bundle of rays whose chief ray PB (or QB) is incident on the surface at the point B at the angle of incidence α. This ray crosses the axis at the point marked L in Fig. 235 and the corresponding refracted ray crosses the

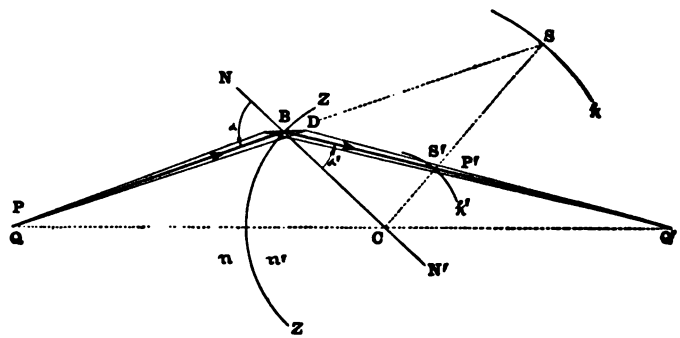


FIG. 236.—Sagittal section of narrow bundle of rays refracted at spherical surface.

axis at L′. One of the principal sections of the bundle of incident rays will be the *meridian section* (§§ 112, 113) made by the plane containing the optical axis and the vertex P (or Q) of the bundle, that is the plane of the paper; whereas the other principal section, called the *sagittal section* (Fig. 236), is made by a plane which intersects the meridian plane at right angles along the chief ray of the bundle. The point G (Fig. 235) is a point on the spherical refracting surface in the meridian section, taken exceedingly close to the point B. Likewise, the point D (Fig. 236) lies on the spherical refracting surface very near to B; but it is contained in the sagittal section and is represented in the diagram as lying slightly above the plane of the paper. The ray PG (Fig. 235) after refraction meets the chief refracted ray at the image-point P′ of the narrow pencil of refracted meridian rays. Similarly, the ray QD (Fig. 236) after refraction will meet the chief refracted ray at the image-point Q′ where the straight line QC intersects this ray, as will be immediately obvious by

supposing that the triangle QBQ′ is revolved around the central line QQ′ as axis through a small angle out from the plane of the paper. Thus, whereas the meridian section of the bundle of refracted rays is contained in the same plane as the meridian section of the bundle of incident rays, the sagittal sections are in two different planes BDQ and BDQ′ (Fig. 236) which intersect each other in a straight line perpendicular to the meridian plane at the point B, that is, in the line BD, which, since the point D is infinitely near to B, may be regarded as a straight line.

185. Formula for Locating the Position of the Image-Point Q′ of a Pencil of Sagittal Rays Refracted at a Spherical Surface.—As was explained (§ 184), the image-point Q′ (Fig. 236) in the sagittal section corresponding to the object-point Q is at the point of intersection of the straight line QC with the chief ray of the bundle of refracted rays. This construction suggests at once a method of obtaining an analytical relation connecting the points Q and Q′; for if the straight line QQ′ is regarded for the time being as the axis of the spherical refracting surface, and if we put $q = \mathrm{BQ}$, $q' = \mathrm{BQ}'$ (where the distances denoted by q, q' are to be reckoned positive or negative according as they are measured from the incidence-point B in the same direction as the light takes along the chief ray or in the opposite direction, respectively), we have merely to write q, q' in place of the symbols l, l' in the formula derived in § 180 in order to obtain the desired relation, namely,

$$\frac{n'}{q'} - \frac{n}{q} = D,$$

where the function denoted here by D is a constant for a given chief ray and is defined by the following expression:

$$D = \frac{n'.\cos a' - n.\cos a}{r} = \frac{n.\sin(a - a')}{r.\sin a'}.$$

Thus having ascertained the path of the chief ray, and knowing the position of the object-point Q, that is, being given

the value of q, we may calculate the value of q' by means of the above formula and thus locate the position of the image-point Q′ of the sagittal section of the bundle of refracted rays.

186. Position of the Image-Point P′ of a Pencil of Meridian Rays Refracted at a Spherical Surface.—The angles of incidence and refraction of the chief ray are denoted by α, α', respectively. Moreover, let θ, θ' (Fig. 235) denote the angles which the chief ray makes with the axis of the spherical refracting surface before and after refraction, respectively; and also let the central angle BCA be denoted by ϕ. Then for a contiguous ray in the meridian section which is incident at the point G very close to the point B, these angles may be denoted by $\alpha+d\alpha$, $\alpha'+d\alpha'$; $\theta+d\theta$, $\theta'+d\theta'$; and $\phi+d\phi$, where $d\alpha$, $d\alpha'$, etc., denote the little increments in the magnitudes of the angles α, α', etc., in passing from the chief ray to an adjacent ray in the meridian section. Now since for the rays PB and PG these angles are connected by the formulæ (§ 180);

$$\alpha=\theta+\phi, \qquad \alpha+d\alpha=\theta+d\theta+\phi+d\phi,$$

we obtain by subtraction:

$$d\alpha=d\theta+d\phi.$$

Around P as center and with radius equal to PB describe the small arc BU which subtends $\angle \mathrm{BPG}=d\theta$; so that we may write:

$$d\theta=-\frac{\text{arc BU}}{p},$$

where $p=\mathrm{BP}$ denotes the distance of the object-point P from the incidence-point B, being reckoned positive or negative exactly in the same way as q in § 185. Now the sides of the little curvilinear triangle BGU may be considered as straight to the degree of approximation with which we are concerned at present, and since the sides of the angle GBU are perpendicular to the sides of the angle of incidence α, so that $\angle \mathrm{GBU}=\alpha$, we obtain:

$$\text{arc BU}=\text{arc GB}.\cos\alpha.$$

Combining this relation with the one above, we have therefore:

$$d\theta = -\frac{\text{arc GB}.\cos a}{p}.$$

Moreover, since $\angle \text{GCB} = d\phi$,

$$d\phi = \frac{\text{arc GB}}{r};$$

and, therefore, by adding this equation to the last and taking account of the relation above, we find:

$$da = \left(\frac{1}{r} - \frac{\cos a}{p}\right).\text{arc GB}. \qquad (1)$$

Similarly, for the corresponding refracted rays BP′ and GP′ which intersect at the image-point P′, for which $\text{BP}' = p'$, we can derive the analogous relation:

$$da' = \left(\frac{1}{r} - \frac{\cos a'}{p'}\right).\text{arc GB}. \qquad (2)$$

Now according to the law of refraction,

$$n.\sin a = n'.\sin a', \qquad n.\sin(a+da) = n'.\sin(a'+da'),$$

and if in the expansions of $\sin(a+da)$ and $\sin(a'+da')$ we write da and da' in place of $\sin da$ and $\sin da'$ and put $\cos da = \cos da' = 1$, as is permissible on account of the smallness of these angles, we may derive the following relation between da and da':

$$n.\cos a.da = n'.\cos a'.da'. \qquad (3)$$

Hence, multiplying equation (1) by $n.\cos a$ and equation (2) by $n'.\cos a'$, and equating the two expressions thus obtained, according to equation (3), we find, after removing the common factor, arc GB, the following formula connecting the ray-intercepts p and p':

$$n'.\cos a'\left(\frac{\cos a'}{p'} - \frac{1}{r}\right) = n.\cos a\left(\frac{\cos a}{p} - \frac{1}{r}\right);$$

which may also be written thus:

$$\frac{n'.\cos^2 a'}{p'} - \frac{n.\cos^2 a}{p} = D,$$

where the symbol D has the same meaning as before in § 185. If we introduce ABBE's differential notation and use the

operator Δ placed in front of a symbol to denote the difference in the value of the magnitude denoted by the symbol before and after refraction, that is, for example, if $\Delta z = z' - z$; then we may write the two formulæ for p and q in the following abbreviated form:

$$\Delta \frac{n}{q} = \Delta \frac{n.\cos^2 a}{p} = D.$$

The position of the image-point P′ of the meridian section of a narrow bundle of rays refracted at a spherical surface may also be quickly ascertained by a simple geometrical

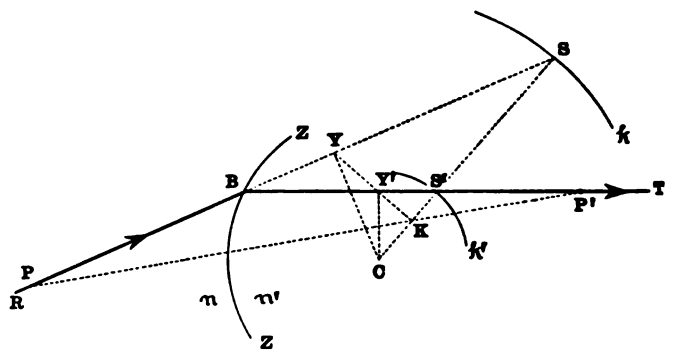


FIG. 237.—Construction of center of perspective (K) with respect to a given ray refracted at a spherical surface.

construction which depends on finding a point K called the *center of perspective*, which bears precisely the same relation to the pair of points P, P′ as the center C of the spherical surface bears to the pair of points Q, Q′ (§ 184); that is, just as the straight line QQ′ must pass through C, so also the straight line PP′ must pass through K. The existence of this point K was first recognized by THOMAS YOUNG (1801). In the diagram (Fig. 237) the chief incident ray is represented by the straight line RB and the chief refracted ray, constructed by the method given in § 175, is represented by the straight line BT. From the center C draw CY and CY′ perpendicular to RB and BT at Y and Y′, respectively. The

point K will be found to lie at the point of intersection of the straight lines YY′ and SS′; and hence if P designates the position of an object-point lying anywhere on the chief incident ray, the corresponding image-point P′ in the meridian section will lie at the point where the straight line PK meets the chief refracted ray. This beautiful construction is exceedingly useful in graphical methods of investigating the imagery in the meridian section along a particular ray. The proof of the construction is not at all difficult, but it cannot be conveniently given here.

187. Measure of the Astigmatism of a Narrow Bundle of Rays.—We have seen that, in general, a narrow homocentric bundle of rays falling obliquely on a spherical refracting surface is transformed into an astigmatic bundle of refracted rays, so that corresponding to a given object-point P (or Q) there will be two so-called image-points P′ and Q′ lying on the refracted chief ray at the points of intersection of the rays of the meridian and sagittal sections, respectively. The interval between these image-points, that is, the segment $P'Q' = q' - p'$ is called the *astigmatic difference*. However, it is more convenient to measure the astigmatism by the difference between the reciprocals of the linear magnitudes p' and q'. If, for example, according to the system of notation introduced in § 106, we put

$$n/p = P, \quad n'/p' = P', \quad n/q = Q, \quad n'/q' = Q',$$

the formulæ derived in §§ 185, 186 may be written as follows:

$$Q' - Q = P'.\cos^2 a' - P.\cos^2 a = D;$$

where, on the assumption that the meter is taken as the unit of length, the magnitudes denoted by the capital letters will all be expressed in terms of the dioptry. The astigmatism of the bundle of refracted rays is measured by $(P' - Q')$. If the bundle of incident rays is homocentric $(Q = P)$, the astigmatism of the bundle of refracted rays will be:

$$P' - Q' = P'.\sin^2 a' - P.\sin^2 a.$$

Accordingly, we see that the astigmatism of a bundle of rays refracted at a spherical surface will vanish provided

$Q=P$ and $P'.\sin^2 a'-P.\sin^2 a=0$; which may happen in two ways, as follows:

(1) If $a'=a=0$, that is, if the chief ray of the narrow bundle meets the refracting surface normally, as, for example, when it is directed along the axis, then no matter where the object-point may lie, the two image-points will coincide. In fact, in case of the axial ray we may put $Q=P=U$, $Q'=P'=U'$, $D=F$, so that the formulæ for the meridian and sagittal sections both reduce in this case to the fundamental equation for the refraction of paraxial rays at a spherical surface, namely, $U'=U+F$.

(2) But for any value of a, we shall have $P'-Q'=0$, that is, $P'.\sin^2 a'=P.\sin^2 a$, provided $P'/n'^2=P/n^2$ or $n'.p'=n.p$. In this case the points designated by P, P' (or Q, Q') are identical with the points S, S' in Figs. 226 to 229. If the vertex of the homocentric bundle of incident rays lies at any point S on the surface of the sphere described around C as center with radius equal to $n'.r/n$, the bundle of refracted rays will likewise be homocentric with its vertex at the corresponding point S' on the surface of the concentric sphere of radius $n.r/n'$ (see § 177).

188. Image-Lines (or Focal Lines) of a Narrow Astigmatic Bundle of Rays.—In all the preceding discussion of the properties of an astigmatic bundle of rays, it cannot have escaped notice that only such rays have been considered as are contained in the two principal sections of the bundle. If there were no other rays to be taken into account besides these, we might say that to each point of the object P (or Q) there corresponded two image-points P' and Q'. But this is by no means a complete or even approximately complete statement of the image-phenomenon in this case; for, indeed, the rays which lie in neither of the two principal sections do, as a matter of fact, constitute by far the greater portion of the total number of rays of the bundle. According to the theorem of STURM (1803–1855), the constitution of a narrow bundle of rays is exhibited in the accompanying diagram

(Fig. 238) called STURM's conoid (§ 113). All the rays of the bundle pass through two very short focal lines or *image-lines* XX and YY which are both perpendicular to the chief ray. The image-line XX which goes through the point of intersection P′ of the meridian rays lies in the plane of the sagittal section; and, similarly, the image-line YY which goes through the point of intersection Q′ of the sagittal rays lies in the

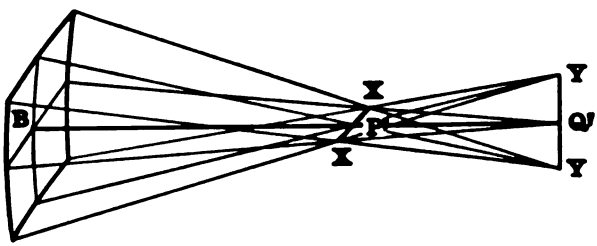


FIG. 238.—STURM's conoid.

plane of the meridian section. Strictly speaking, this theorem can be regarded as representing the actual facts only on the assumption that the bundle of rays is infinitely thin; and on this assumption the entire bundle may be conceived as generated by a slight rotation either of the meridian section around the image-line YY as axis, whereby the point P′ will trace the image-line XX, or of the sagittal section around the image-line XX as axis, whereby the point Q′ will trace the image-line YY. Thus, according to STURM's theorem, with an object-point P (or Q) lying on the chief ray of an infinitely narrow bundle of incident rays there are associated two exceedingly tiny image-lines lying in the principal sections of the bundle of refracted rays at right angles to the chief ray. Not only as to the orientation of the image-lines of STURM, but as to their practical, nay, even as to their mathematical existence, there has been much controversy, but we cannot enter into this discussion here. In spite of its limitations and admittedly imperfect representation, STURM's conoid remains a very useful preliminary mode of conception of the character of a narrow astigmatic bundle

of rays. The only proper way of arriving at a more accurate knowledge of the constitution of a bundle of light-rays is by the aid of the powerful methods of the infinitesimal geometry. Mathematical investigations of this kind have been pursued with great skill by GULLSTRAND whose writings contained in a series of published papers and treatises dating from about 1890 have extended the domain of theoretical optics far beyond the narrow limits imposed upon it by GAUSS and the earlier writers on this subject.

189. The Astigmatic Image-Surfaces.—Thus, the effect of astigmatism is that the rays of a narrow oblique bundle, instead of being brought to a focus at a single point, pass through two small focal lines at right angles to the path of

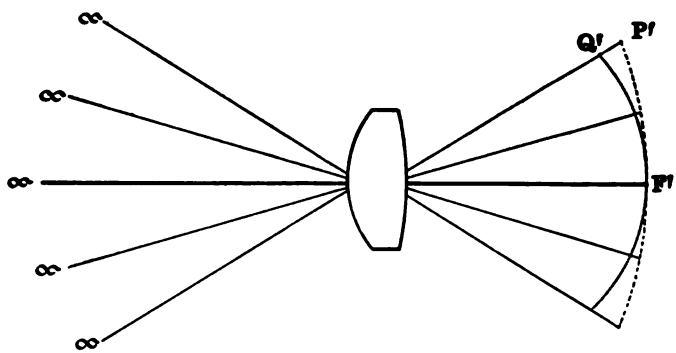


FIG. 239.—Astigmatic image-surfaces.

the chief ray in the image-space. If the chief rays proceeding from the various object-points lying in a meridian plane of a symmetrical optical instrument are constructed, and if along each of these rays the positions of the image-points P′, Q′ of the pencils of meridian and sagittal rays are determined, the loci of these points will be two curved lines, both symmetrical with respect to the axis, which touch each other at their common vertex on the axis. In the diagram (Fig. 239) the object is supposed to be infinitely distant, as, for example, in the case of a landscape photographic lens. The contin-

uous curved line represents the locus of the points of intersection of the sagittal rays, whereas the dotted curve represents the locus of the points of intersection of the meridian rays. These curved lines are the traces in the meridian plane of the two *astigmatic image surfaces* which are generated by revolving the figure around the axis of symmetry. The two image-surfaces which correspond to a definite transversal plane in the object-space, and which are the loci of the most sharply defined images of object-points lying in this plane, are not to be confused with the two sheets of the caustic surface of a wide-angle bundle of rays emanating from a single point of an object (§ 183). The focal lines of the narrow pencils of meridian rays lie on one of these surfaces and the focal lines of the narrow pencils of sagittal rays lie on the other surface. The positions and forms of the image-surfaces will depend essentially on the place of the stop; for it is evident that if the stop is shifted to a different place, the chief ray of each bundle (§§ 140, 184) will be a different ray, and the points P′ and Q′ will all occupy entirely different positions. If a curved screen could be exactly adjusted to fit one of the image-surfaces, a fairly sharp image of the object might be focused on it, but not only would the image be curved instead of flat, but there would also be a certain *astigmatic deformation* due to the fact that each point of the object would be reproduced not by a point but by a little focal line, as has been explained. Between the two image-points P′ and Q′ on each chief ray there lies a certain approximately circular cross-section of the narrow astigmatic bundle known (§ 113) as the "circle of least confusion," and the locus of the centers of these circles will lie on a third surface intermediate between the other two, which is sometimes taken as a kind of average or compromise image-surface.

There can be no doubt that astigmatism of oblique bundles is responsible for serious defects in the image produced by an optical instrument, and much pains has been bestowed on trying to remedy this fault as far as possible. Fortunately,

the possibility of abolishing astigmatism of this kind, that is, of making the two image-surfaces coincide in a single surface, is afforded by the fact that the astigmatic difference (§ 187) is opposite in sign according as the refracting surface is convergent or divergent. For example, Fig. 240 shows graphically the opposite effects of a convergent and a divergent spherical refracting surface under otherwise equal conditions. The two curves on the left-hand side relate to the convergent system, and the two curves on the right-hand side relate to the divergent system; and we see that not only are the curvatures opposite in the two cases, but the relative positions of the curves are different. It will not be difficult to understand that it may be possible, by suitable choice of the radii of the refracting surfaces and of their distances apart and also of the position of the stop, to design a system which will be free from astigmatism at any rate for a certain zone of the lens; so that, although we may not be able to make the two astigmatic image-curves coincide absolutely throughout their entire extent, we may contrive so that the two curves are nowhere very far apart, while at one point, corresponding to the corrected zone, they actually intersect each other.

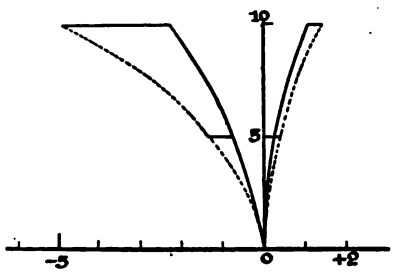


FIG. 240.—Astigmatism of convergent spherical refracting surface (plotted on the left) and astigmatism of divergent spherical refracting surface (plotted on the right).

190. Curvature of the Image —Now let us suppose that the astigmatism of oblique bundles has been completely abolished for a certain angular extent of the field of view, so that at last there is strict point-to-point correspondence by means of narrow bundles of rays between object and image. The two image-surfaces have thus been merged into one, and

over this surface, within the assigned limits, the definition of the image is clear-cut and distinct. There still remains, however, another trouble due to the fact that the image is curved and not flat; consequently, if the image is received on a plane focusing screen, only those parts of the stigmatic image which lie in the plane of the screen will be in focus (Fig. 241), whereas the rest of the image on the screen will be blurred.

Now this error of the curvature of the image cannot be

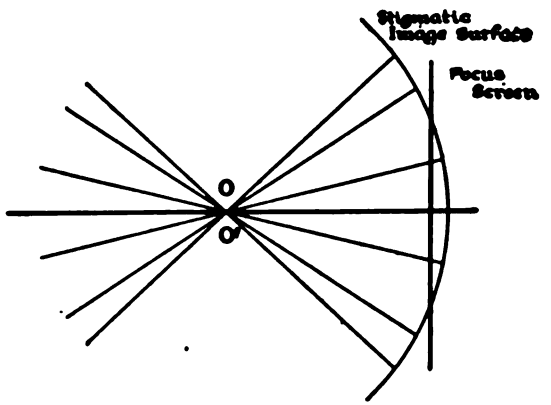


FIG. 241.—Curvature of stigmatic image.

overcome by employing methods similar to those above described for the abolition of astigmatism. For the correction of the latter error the particular kinds of glass of which the lenses were made were not essential; whereas with unsuitable kinds of glass there is no choice of the radii, thicknesses, etc., which will yield an image which is at the same time stigmatic and flat. This fact was well known to PETZVAL (1807–1891). PETZVAL'S formula (published in 1843) for the abolition of the curvature of a stigmatic image produced by a system of infinitely thin lenses in contact with each other is

$$\Sigma \frac{F_1}{n_1} = 0,$$

where F_i denotes the refracting power and n_i denotes the index of refraction of the ith lens of the system. This formula is equivalent also to the following statement: The curvature of the stigmatic image of an infinitely distant object in a system of lenses whose total thickness is negligible is equal to

$$\Sigma\left\{\frac{1}{n}\Sigma\ (\text{refracting powers of all lenses of index } n)\right\}$$

The general principal of this equation was discovered by AIRY and was given by CODDINGTON in his treatise published in 1829. SEIDEL pointed out that the two faults of astigmatism and curvature could not both be corrected at the same time unless some of the convex lenses of the system were made of more highly refracting glass than the concave lenses. Now with the varieties of glass which were available before the production of the modern Jena glass, this requirement was directly opposed to the condition of achromatism, and as the latter error was considered more serious than the curvature-error, the earlier lens-designers made no attempt to obtain a flat stigmatic image. But with the new kinds of glass now at our disposal, it is possible to design the optical system so that not only is the astigmatism corrected for a certain zone, as explained in § 189, but the point of intersection of the two image-lines lies in the same transversal plane as the axial point where the two image-lines touch each other (Fig. 242). Accordingly, we may say that for this zone the image is both flat and stigmatic. The construction of modern photographic lenses which are practically free from these spherical errors is an almost unsurpassed triumph of human ingenuity.

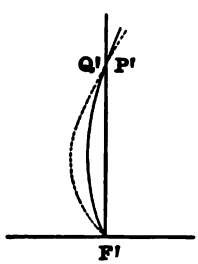


FIG. 242. — Stigmatic image in transversal focal plane for a given zone of optical system.

191. Coma.—Astigmatism implies that the bundles of rays concerned in producing the image are very narrow, and

this means that the diameter of the stop is very small. But the validity of the assumptions which are at the foundation of geometrical optics begins to be called in question in the case of narrow bundles of rays, as was pointed out in § 175; so that we must be careful here not to push our conclusions too far. As a matter of fact, in various optical instruments and particularly in some modern types of photographic lenses, the diameter of the stop is by no means small and the

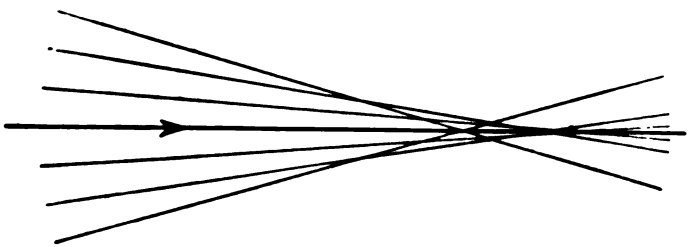


FIG. 243.—Symmetrical character of sagittal section.

field of view is extensive. The spherical aberrations which are encountered in an optical system of this kind are of an exceedingly complicated nature which cannot be described here in detail.

A bundle of rays of finite aperture emanating from a point outside the optical axis will show aberrations of a general character similar to the aberrations along the axis of a direct bundle of rays (§ 178). But the effects in the two principal sections of the bundle will be very different from each other; because, whereas the rays in the sagittal section, being symmetrically situated on opposite sides of the meridian plane, are therefore symmetrical with respect to the chief ray, as represented in Fig. 243, there will, in general, be a complete absence of symmetry in the meridian section (Fig. 244). The image (if indeed we may continue to use this term) of an extra-axial object-point under such circumstances will be at best an element of one or other of the two sheets of the caustic surface. Usually, however, what is called the image

is the light-effect as obtained on a focusing screen placed at right angles to the axis at the place where the central parts of the object are best delineated. The appearance on the screen may be described as a kind of balloon-shaped flare of light,

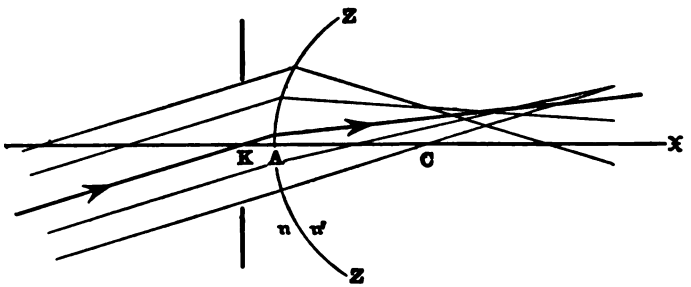


FIG. 244.—Unsymmetrical character of meridian section, giving rise to coma.

with a bright nucleus growing fainter as it expands in some cases towards, in other cases away from, the axis. This defect of the image is known to practical opticians as side-flare or *coma* (from the Greek word meaning "hair" from which the word "comet" is likewise indirectly derived). The definition in the outer parts of the field of the object-glass of a telescope depends on the removal of this error; and this applies also to the case of a wide-angle photographic lens. The only way to obtain a really clear and accurate conception of this important spherical aberration is to study the forms of the two sheets of the caustic surface. Generally speaking, we may say that the convergence of wide-angle bundles of rays will be better in the case of an optical system which has been corrected for astigmatism, but even then there will be lack of symmetry in all the sections of a bundle of rays except in the sagittal section. If the slope of the chief ray is comparatively slight, although not negligible, the condition of a sharp focus is equivalent to ABBE's sine-condition (§ 182). But for greater inclinations of the chief rays, it will generally be necessary to resort to the exact methods of trigonometri-

cal calculation of the ray-paths in order to determine the nature and degree of the convergence.

192. Distortion; Condition of Orthoscopy.—Let us assume that the system has been corrected for both astigmatism and curvature of the image, in the sense explained in § 190; so that by means of narrow bundles of rays a flat stigmatic image is obtained of a plane object placed at right angles to the axis. The next question will be to inquire whether the image is a faithful reproduction of the object or whether it is distorted. If the image in the "screen-plane" (§ 134) is geometrically similar to the object-relief projected from the center of the entrance-pupil on the "focus-plane" (§ 141), then we may say that the optical system is *orthoscopic* or free from distortion.

The dissimilarity which may exist between an object and its image is a fault of an essentially different kind from those which have been previously considered, and there is no in-

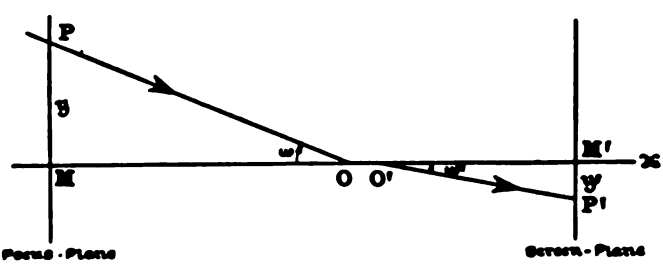


FIG. 245.—Condition of orthoscopy (freedom from distortion).

timate connection between this defect and the others. Here we are not concerned so much with the quality and definition of the image on the screen as with the positions of the points where the chief rays cross the screen-plane. The positions of these representative points will not be altered by reducing the stop-opening (§§ 141, 142); and accordingly the image in the screen-plane is to be regarded merely as a central projection on this plane along the chief rays proceeding from the center of the exit-pupil.

In the diagram (Fig. 245) the centers of the entrance-pupil and exit-pupil of the optical system are designated by O and O′. The straight lines PO, P′O′ represent the path of a chief ray which crosses the focus-plane in the object-space at the point P and the screen-plane in the image-space at the point P′. If $y = MP$, $y' = M'P'$ denote the distances of P, P′ from the axis, then the condition that the image in the screen-plane shall be similar to the projected object in the focus-

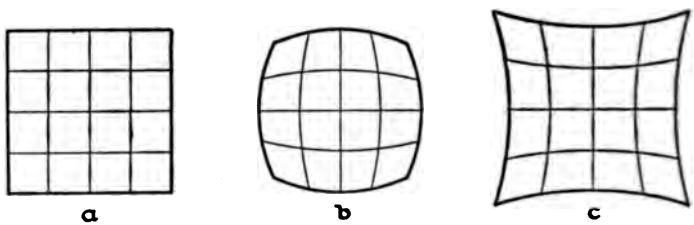


FIG. 246.—Object (*a*) reproduced by image (*b*) barrel-shaped distortion or by image (*c*) cushion-shaped distortion.

plane, that is, the condition of orthoscopy (freedom from distortion) is that the ratio y'/y shall have a constant value for all values of y within the limits of the field of view. If, on the contrary, this is not the case, and if the ratio y'/y is variable for different values of y, then the image will be distorted; and this distortion will be one of two kinds according as the ratio y'/y increases or decreases with increase of y. For example, if the object is in the form of a square, as shown in Fig. 246, *a*, then, on the supposition that y'/y decreases as y increases the image of the diagonal will be shortened relatively more than the image of a side of the square, and the square will be reproduced by a curvilinear figure with convex sides as shown in Fig. 246, *b;* this is the case of *barrel-shaped distortion*, as it is called. On the other hand, if the ratio y'/y increases in proportion as the object-point is taken farther and farther from the axis, we have the opposite type known as *cushion-shaped distortion* (Fig. 246, *c*).

If in Fig. 245 we put $OM = z$, $O'M' = z'$, $\angle MOP = \omega$,

$\angle M'O'P' = \omega'$, the condition of orthoscopy may be expressed as follows:

$$\frac{y'}{y} = \frac{z'.\tan\omega'}{z.\tan\omega} = \text{constant};$$

and if we assume, as has been tacitly assumed in the preceding discussion, that the chief rays all pass through the pupil-centers O, O', so that the abscissæ denoted by z, z' have the same values for all distances of the object-point P from the axis, then we derive at once AIRY's *tangent-condition* of orthoscopy, namely, $\tan\omega' : \tan\omega = \text{constant}$. But although a chief ray must pass through the center of the aperture-stop (§ 140), it will not pass through the centers of the pupils unless the latter are free from spherical aberration. The constancy of the tangent-ratio by itself is not a sufficient condition for orthoscopy; in addition, the spherical aberration must be abolished with respect to the centers of the pupils.

If the optical system is symmetrical with respect to an interior aperture-stop, the tangent-condition will be immediately satisfied, because on account of the symmetry of the two halves of the system, every chief ray will issue in exactly the same direction as it had on entering, and therefore $\tan\theta : \tan\theta' = 1$. Accordingly, if a "symmetric doublet" of this kind is spherically corrected with respect to the center of the aperture-stop, it will give an image which will be free from distortion.

193. Seidel's Theory of the Five Aberrations.—In the theory of optical imagery which was developed according to general laws first by GAUSS (§ 119) in his famous *Dioptrische Untersuchungen* published in 1841, the fundamental assumption is that the effective rays are all comprised within a narrow cylindrical region of space immediately surrounding the optical axis; this region being more explicitly defined by the condition that a paraxial ray is one for which the angle of incidence (α) and the slope-angle (θ), in the case of each refraction or reflection, are both relatively so minute that the

powers of these angles higher than the first can be neglected (§ 63). Evidently, therefore, GAUSS's theory is applicable only to optical systems of exceedingly small aperture and limited extent of field of view. But with the development of modern optical instruments and especially with the increase of both aperture and field demanded for certain types of photographic lenses, it became necessary to take account of rays which lie far beyond the narrow confines of the central or paraxial rays. Long prior to the time of GAUSS important contributions to the theory of spherical aberrations had been made in connection with certain more or less special problems; but the first successful attempt to extend GAUSS's theory in a general way by taking account of the terms of higher orders of smallness was made by SEIDEL (1821–1896) in a remarkable series of papers published between the years 1852 and 1856 in the *Astronomische Nachrichten*. SEIDEL's method consisted in tracing the path of the ray through the centered system of spherical refracting surfaces and in developing the trigonometrical expressions in series of ascending powers which were finally simplified by neglecting all terms above the third order. If the ray-parameters are regarded as magnitudes of the first order of smallness, it is easy to show that on account of the symmetry around the optical axis these series-developments can contain only terms of the odd orders of smallness; so that in SEIDEL's theory the terms neglected are of the fifth and higher orders. It is impossible to describe here in detail the elegant mathematical treatment by which SEIDEL was enabled to arrive at his final results; suffice it to say, that he obtained a system of formulæ from which it was possible to ascertain the influence both of the aperture and the field of view on the perfection of the image. In SEIDEL's formulæ the aberrations of the ray, that is, its deviations from the path prescribed by GAUSS's theory, are expressed by five different sums, denoted by S_1, S_2, S_3, S_4, and S_5, which depend only on the constants of the optical system and the position of the

object-point, and which are, in fact, the cöefficients of the various terms in the equations. The condition that there shall be no aberration demands that all of these five sums shall vanish simultaneously, that is,

$$S_1 = S_2 = S_3 = S_4 = S_5 = 0.$$

If, on the other hand, these conditions are not satisfied, the image yielded by the lens-system will not be faultless; and therefore it will not be without interest to inquire more particularly into the separate influence of each of these five expressions which occur in SEIDEL'S formulæ.

Thus, for example, if the optical system is designed so that $S_1 = 0$, then there will be no spherical aberration at the center of the field (§ 178) for the given position of the axial object-point. And if not only $S_1 = 0$ but also $S_2 = 0$, then there will be no coma (§ 191). The condition $S_2 = 0$ means also that ABBE'S sine-condition (§ 182) will also be satisfied, so that the image of the parts of the object in the immediate vicinity of the axis is sharply defined.

But even when we have $S_1 = S_2 = 0$, the optical system will, in general, still be affected by astigmatism of oblique rays (§ 184), so that an object-point lying at some little distance from the axis will not be reproduced by an image-point but at best by two short focal lines at different distances from the lens-system and directed approximately at right angles to each other. Moreover if the distance of the object-point from the axis is varied, the positions of these two focal lines will vary also both with respect to their distance from the lens-system and with respect to their mutual distance apart. In other words, when both S_1 and S_2 vanish, then, in general, there is no unique image of a transversal object-plane, but this latter may be said to be reproduced by two so-called image-surfaces (§ 189) which are surfaces of revolution around the optical axis and which unite and touch each other at the point where the axis crosses them. The expressions for the curvatures of these surfaces at this common point of tangency are given by SEIDEL'S sums S_3 and

S_4; so that if also $S_3 - S_4 = 0$, the two image surfaces will coalesce and now the image of the plane object will be sharply defined, that is, stigmatic, although it will usually still be curved. But if also $S_3 = S_4 = 0$, the image will be both plane and stigmatic. However, it may still show unequal magnifications toward the margin, which means that there is distortion (§ 192). This last error will be abolished provided $S_5 = 0$; and now the image may be said to be ideal inasmuch as it is flat and sharply defined not only in the center but out toward the edges and is at the same time a faithful reproduction of the plane object.

To attempt to derive SEIDEL'S actual formulæ or even to discuss the equations would be entirely beyond the scope

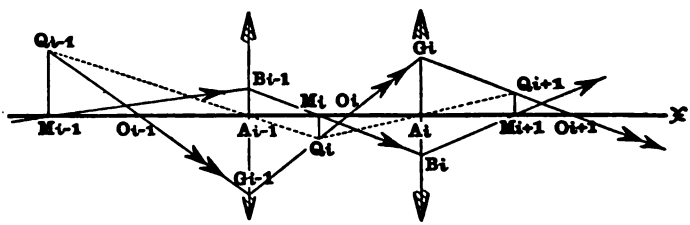


FIG. 247.—Diagram representing the $(i-1)$th and ith lenses of a system of infinitely thin lenses.

of this volume. But it may be convenient to insert here without proof the expressions of SEIDEL'S five sums for the comparatively simple case of an optical system considered as composed of a series of m infinitely thin lenses each surrounded by air.

Let A_i (Fig. 247) designate the point where the optical axis crosses the ith lens of the system, the symbol i being employed to denote any integer from 1 to m; and let us consider two paraxial rays which traverse the optical system, one of which emanating from the axial object-point M_1 ($A_1M_1 = u_1$) and meeting the first lens at a point B_1 such that $A_1B_1 = h_1$, crosses the axis after passing through the $(i-1)$th lens at a point M_i ($A_iM_i = u_i$) and meets the ith lens at a point B_i

such that $A_iB_i = h_i$ whereas the other ray, which emanates from an extra-axial object-point and which in the object-space passes through the center O_1 of the entrance-pupil (§ 139) of the system ($A_1O_1 = s_1$) and meets the first lens at a point G_1 such that $g_1 = A_1G_1$, crosses the axis after passing through the $(i-1)$th lens at a point O_i $A_iO_i = s_i$) and meets the ith lens at a point G_i such that $A_iG_i = g_i$. Then if we put

$$U_i = 1/u_i, \qquad S_i = 1/s_i,$$

it may easily be shown that

$$h_i(U_i + F_i) = h_{i+1}.U_{i+1},$$
$$g_i(S_i + F_i) = g_{i+1}.S_{i+1};$$

where F_i denotes the refracting power of the ith lens. Now if n_i denotes the index of refraction of the ith lens and if R_i denotes the curvature of the first surface of this lens; and if, further, for the sake of brevity, the symbols A_i, B_i, C_i, D_i, and E_i are introduced to denote the following functions of n_i, F_i, R_i, U_i and S_i, namely:

$$A_i = \frac{n_i+2}{n_i} F_iR_i^2 - \left\{ \frac{4(n_i+1)}{n_i} U_i + \frac{2n_i+1}{n_i-1} F_i \right\} F_iR_i + \frac{3n_i+2}{n_i} F_iU_i^2 + \frac{3n_i+1}{n_i-1} F_i^2U_i + \left(\frac{n_i}{n_i-1}\right)^2 F_i^3;$$

$$B_i = \frac{n_i+2}{n_i} F_iR_i^2 - \left\{ \frac{n_i+1}{n_i}(3U_i+S_i) + \frac{2n_i+1}{n_i-1} F_i \right\} F_iR_i + \frac{n_i+1}{n_i} F_iU_i^2 + \frac{2n_i+1}{n_i} F_iU_iS_i + \frac{2n_i+1}{n_i-1} F_i^2U_i + \frac{n_i}{n_i-1} F_i^2S_i + \left(\frac{n_i}{n_i-1}\right)^2 F_i^3;$$

$$C_i = \frac{3(n_i+2)}{n_i} F_iR_i^2 - \left\{ \frac{6(n_i+1)}{n_i} (U_i+S_i) + \frac{3(2n_i+1)}{n_i-1} F_i \right\} F_iR_i + \frac{1}{n_i} F_iU_i^2 + \frac{2(3n_i+2)}{n_i} F_iU_iS_i + \frac{3n_i+1}{n_i} F_iS_i^2 + \frac{3(n_i+1)}{n_i-1} F_i^2U_i + \frac{6n_i}{n_i-1} F_i^2S_i + 3\left(\frac{n_i}{n_i-1}\right)^2 F_i^3;$$

$$D_i=\frac{n_i+2}{n_i}F_iR_i^2-\left\{\frac{2(n_i+1)}{n_i}(U_i+S_i)+\frac{2n_i+1}{n_i-1}F_i\right\}F_iR_i+\frac{1}{n_i}F_iU_i^2$$
$$+2F_iU_iS_i+\frac{n_i+1}{n_i}F_iS_i^2+\frac{n_i+1}{n_i-1}F_i^2U_i$$
$$+\frac{2n_i}{n_i-1}F_i^2S_i+\left(\frac{n_i}{n_i-1}\right)^2F_i^3;$$

$$E_i=\frac{n_i+2}{n_i}F_iR_i^2-\left\{\frac{n_i+1}{n_i}(U_i+3S_i)+\frac{2n_i+1}{n_i-1}F_i\right\}F_iR_i+\frac{3n_i+1}{n_i}F_iS_i^2$$
$$+\frac{1}{n_i}F_iU_iS_i+\frac{1}{n_i-1}F_i^2U_i+\frac{3n_i}{n_i-1}F_i^2S_i+\left(\frac{n_i}{n_i-1}\right)^2F_i^3;$$

then SEIDEL's formulæ for the spherical errors of a system of m infinitely thin lenses may be expressed as follows:

$$S_1=\sum_{i=1}^{i=m}\left(\frac{h_i}{h_1}\right)^4A_i; \qquad S_2=\sum_{i=1}^{i=m}\left(\frac{h_i}{h_1}\right)^3\frac{g_i}{g_1}B_i;$$

$$S_3=\sum_{i=1}^{i=m}\left(\frac{h_i}{h_1}\cdot\frac{g_i}{g_1}\right)^2C_i; \qquad S_4=\sum_{i=1}^{i=m}\left(\frac{h_i}{h_1}\cdot\frac{g_i}{g_1}\right)^2D_i;$$

$$S_5=\sum_{i=1}^{i=m}\frac{h_i}{h_1}\left(\frac{g_i}{g_1}\right)^3E_i.$$

The greatest practical value of these formulæ is to guide the optician to a correct basis for the design of his instrument and to supply him, so to speak, with a starting point for a trigonometrical calculation of the optical system which he aims to achieve. But the reader who wishes to pursue this subject further will find it necessary to consult the more advanced treatises on applied optics.

PROBLEMS

1. If L, L′ designate the points where a ray crosses the axis of a spherical refracting surface before and after refraction, respectively, and if C designates the center of the surface, show that

$$\frac{n}{c'}-\frac{n'}{c}=\frac{n'-n}{r}\,\frac{\cos\frac{\theta+\theta'}{2}}{\cos\frac{a+a'}{2}},$$

where $c=\mathrm{CL}$, $c'=\mathrm{CL}'$, a, a' denote the angles of incidence and refraction, θ, θ' denote the slope-angles of the ray before and after refraction, r denotes the radius of the surface, and n, n' denote the indices of refraction. Also, show that

$$\frac{c'+r}{c+r}=\frac{\sin\theta\cos\frac{a'+\theta'}{2}}{\sin\theta'\cos\frac{a+\theta}{2}}.$$

2. A ray parallel to the axis meets the first surface of a glass lens (index 1.5) at a height of 5 cm. above the axis, and after emerging from the lens crosses the axis at a point L′. The thickness of the lens is 1 cm. Determine the aberration F′L′, where F′ designates the position of the second focal point, for each of the following cases: (*a*) First surface of lens is plane and radius of curved surface is 50 cm.; (*b*) Second surface of lens is plane and radius of curved surface is 50 cm.; and (*c*) Lens is symmetric, radius of each surface being 100 cm.

Ans. (*a*) $f=\pm 100$ cm., $\mathrm{F'L'}=\mp 1.13$ cm.; (*b*) $f=\pm 100$ cm., $\mathrm{F'L'}=\mp 0.29$ cm.; (*c*) $f=\pm 100.17$ cm., $\mathrm{F'L'}=\mp 0.42$ cm.; where in each case the upper signs apply to positive lens and the lower signs apply to negative lens.

3. An incident ray crosses the axis of a lens at an angle θ_1 and meets the first surface at a point B_1, the angle of incidence being a_1; the slope of the refracted ray B_1B_2, which

meets the second surface at the point B_2 is θ_2, and the angle of incidence at this surface is α_2. If the radii of the surfaces are denoted by r_1 and r_2, show that

$$B_1B_2 = \frac{r_2.\sin(\alpha_2 - \theta_2) - r_1.\sin(\alpha_1 - \theta_1)}{\sin\theta_2}.$$

4. The chief ray of a narrow bundle of parallel rays is incident on a spherical mirror of radius 32 cm. at an angle of 60°. Find the distance between the two image-points P′ and Q′ of the bundle of reflected rays. Ans. 24 cm.

5. The chief ray of a narrow bundle of parallel rays is incident on a spherical mirror of radius r at a point B, the angle of incidence being 60°. Determine the positions of the image-points P′ and Q′. Ans. $BP' = r/4$, $BQ' = r$.

6. A narrow bundle of parallel rays in air is refracted at a spherical surface of radius r into a medium whose index of refraction is $\sqrt{3}$. If the angle of incidence is 60°, find the positions of the image-points P′ and Q′.

Ans. $p' = 3r\sqrt{3}/4$, $q' = r\sqrt{3}$.

7. A narrow bundle of parallel rays is incident on a spherical refracting surface at an angle of 60°. If the meridian rays are converged to a focus at a point P′ lying on the surface of the sphere, show that the angle of refraction of the chief ray is equal to the complement of the critical angle of the two media.

8. The radius of each of the two surfaces of an infinitely thin double convex lens is 8 inches, and the index of refraction is equal to $\sqrt{3}$. The chief ray of a narrow bundle of parallel rays inclined to the axis at an angle of 60° passes through the optical center of the lens. Find the positions of the foci of the meridian and sagittal rays.

Ans. The focal point of the meridian rays is 1 inch and that of the sagittal rays is 4 inches from the optical center.

9. If in YOUNG's construction of a ray refracted at a spherical surface (§ 176) a semi-circle is described on the incidence-radius BC as diameter intersecting the incident and refracted rays in the points Y, Y′, respectively, show that the straight

line YY′ is perpendicular to the straight line CS. The point K where the straight lines YY′ and CS meet is the center of perspective of the range of object-points lying on the incident ray and the corresponding range of meridian image-points lying on the refracted ray (see § 186). Show that

$$\mathrm{CK} = \frac{n.r.\sin^2 a}{n'},$$

and that

$$\tan \angle \mathrm{BKC} = \tan a + \tan a'.$$

10. If the chief ray of a narrow homocentric bundle of rays is incident on a plane refracting surface at a point B, and if a, a' denote the angles of incidence and refraction, show that

$$\mathrm{BP}' = \frac{n' \cos^2 a'}{n \cos^2 a} . \mathrm{BP}, \qquad \mathrm{BQ}' = \frac{n'}{n} . \mathrm{BQ},$$

where P (or Q) designates the position of the vertex of the incident rays and P′ and Q′ designate the positions of the image-points of the meridian and sagittal rays, respectively.

11. In the preceding problem show that the straight line QQ′ is perpendicular to the plane refracting surface.

12. The position of the image-point P′ of a pencil of meridian rays refracted at a plane surface may be constructed as follows: Through the object-point P (or Q) draw PQ′ perpendicular to the refracting plane and meeting the chief refracted ray in Q′; and from P and Q′ draw PX and Q′Y perpendicular to the incidence-normal at X and Y, respectively. Draw XG perpendicular to the chief incident ray at G and YG′ perpendicular to the corresponding refracted ray at G′. Then the straight line PP′ drawn parallel to GG′ will intersect the chief refracted ray in the required point P′. Using the result of No. 10 above, show that this construction is correct.

13. The chief ray RB of a narrow pencil of sagittal rays meets a spherical refracting surface at the point B and is refracted in the direction BT. Through the center C draw CV

parallel to BT meeting BR in V and CV′ parallel to BR meeting BT in V′. If Q, Q′ designate the positions of the points of intersection of the sagittal rays before and after refraction, respectively, and if $BQ=q$, $BQ'=q'$, show that

$$\frac{BV}{q}+\frac{BV'}{q'}=1,$$

and that

$$VQ.V'Q'=VB.V'B.$$

(Compare this last result with the Newtonian formula for refraction of paraxial rays at a spherical surface, viz., $xx'=ff'$.)

14. The chief ray RB of a narrow pencil of meridian rays meets a spherical refracting surface at the point B, and is refracted in the direction BT. Through the center of perspective K (see § 186; see also problem No. 9 above) draw KU parallel to BT meeting BR in U and KU′ parallel to BR meeting BT in U′. If the positions of the points of intersection of the meridian rays before and after refraction are designated by P and P′, respectively, and if $BP=p$, $BP'=p'$, show that

$$\frac{BU}{p}+\frac{BU'}{p'}=1,$$

and that

$$UP.U'P'=UB.U'B.$$

(Compare this result with that of the preceding problem.)

15. If J, J′ designate the positions of the aplanatic points of a spherical refracting surface, and if θ, θ' denote the slopes of the incident and refracted rays BJ, BJ′, respectively, show that

$$\frac{\sin\theta}{\sin\theta'}=\frac{n'}{n}\cdot y,$$

where y denotes the magnification-ratio for paraxial rays.

16. A. Steinheil's so-called "periscope" photographic lens is composed of two equal simple meniscus lenses, both of crown glass, separated from each other with a small stop midway between. The data of the system, as given in Von

ROHR's *Theorie und Geschichte des photographischen Objektivs* (Berlin, 1899), p. 288, are as follows:

Indices: $n_1 = n_3 = n_5 = 1$; $n_2 = n_4 = 1.5233$

Radii: $r_1 = -r_4 = +17.5$ mm.; $r_2 = -r_3 = +20.8$ mm.

Thicknesses: $d_1 = d_3 = +1.3$ mm.; $d_2 = 12.6$ mm.

Distance of center of stop from second vertex of first lens $= +6.3$ mm.; diameter of stop $= 2.38$ mm.; diameter of each lens $= 11.32$ mm.

Employing the above data, determine (1) the position and size of the entrance-pupil, (2) the angular extent of the field, (3) the position of the second focal point F'; and (4) the point where an edge-ray directed towards a point in the circumference of the entrance-pupil and parallel to the axis crosses the axis after emerging from the system.

Ans. (1) Distance of center of entrance-pupil from second vertex of first lens is $+6.45$ mm.; diameter of entrance-pupil is 2.53 mm. (2) The angular extent of the field is nearly 90°. (3) Distance of F' from last surface is $A_4F' = +90.946$ mm. (4) The edge-ray crosses the axis at a distance $A_4L_4 = +90.432$ mm.

17. The abscissæ of the points M_k, M_{k+1} where a paraxial ray crosses the axis of a centered system of m spherical refracting surfaces before and after refraction at the kth surface are denoted by $u_k = A_kM_k$, $u_k' = A_kM_{k+1}$. If the ray proceeds in the first medium of index n_1 in a direction parallel to the axis, it may be shown (*cf.* problems Nos. 16 and 17, end of Chapter X) that the primary focal length of the system is given by the formula

$$f = \frac{U_2 . U_3 \ldots U_m}{U_1' . U_2' \ldots U_m'}, (U_1 = 0),$$

where $U_k = n_k/u_k$, $U_k' = n_{k+1}/u_k'$. Having calculated the path of the paraxial ray in the preceding problem, employ the above formula to determine the focal length of STEINHEIL's "periscope." Ans. $f = +98.696$ mm.

18. The path of a chief ray which in traversing the air-space between the two lenses of STEINHEIL's "periscope"

(see No. 16) goes through the center of the stop will be symmetrical with respect to the two parts of the optical system, so that for such a ray we must have:

$$c_4' = -c_1, \quad c_4 = -c_1', \quad c_3' = -c_2, \quad c_3 = -c_2';$$
$$a_4 = a_1', \quad a_4' = a_1, \quad a_3 = a_2', \quad a_3' = a_2;$$
$$\theta_5 = \theta_1, \quad \theta_4 = \theta_2.$$

Show that if $\theta_3 = -30°$ for a ray which goes through the stop-center, the ray must have been directed initially at a slope-angle $\theta_1 = -28°\ 2'\ 54.43''$ towards a point L_1 on the axis whose distance from the second vertex of the first lens is $A_2L_1 = +6.563$ mm.

19. The astigmatism of a narrow bundle of rays refracted through a centered system of spherical surfaces may be computed logarithmically by means of the following recurrent formulæ:

$$D_k = \frac{n_k.\sin(a_k - a_k')}{r_k.\sin a_k'},$$

$$h_k = r_k.\sin(a_k - \theta_k), \quad t_k = \frac{h_{k+1} - h_k}{n_{k+1}.\sin\theta_{k+1}};$$

Sagittal Section

$$Q_k' = Q_k + D_k, \qquad Q_{k+1} = \frac{Q_k'}{1 - t_k.Q_k'};$$

Meridian Section

$$P_k' = \frac{P_k.\cos^2 a_k + D_k}{\cos^2 a_k'}, \quad P_{k+1} = \frac{P_k'}{1 - t_k.P_k'};$$

where the symbols a, a', θ, n and r have their usual meanings and where P, P' and Q, Q' and D are the magnitudes defined in §§ 186 and 184. The calculations according to these formulæ will be considerably simplified in the case of a chief ray which traverses a system like STEINHEIL's "periscope" (see No. 16) which is symmetric with respect to the stop-center. For example, for this particular system we can write for a chief ray:

$$D_4=D_1,\quad D_3=D_2,\quad h_4=-h_1,\quad h_3=-h_2,\quad t_3=t_1;$$
$$a_1-a_1'=-(a_4-a_4'),\quad a_2-a_2'=-(a_3-a_3'),$$
$$a_1-\theta_1=a_4'-\theta_5=a_4-\theta_4,\quad a_2-\theta_2=a_3'-\theta_4=a_3-\theta_3.$$

Apply the above formulæ to the optical system of problem No. 16 to calculate the astigmatic difference (§ 186) of a narrow bundle of emergent rays whose chief ray is the ray whose path was determined in problem No. 18; assuming that the bundle of incident rays was cylindrical, that is, $P_1=Q_1=0$.

Ans. $p_4'-q_4'=+4.849$ mm.

20. Using SEIDEL'S formulæ as given in § 193, show that the condition that an infinitely thin lens surrounded by air, and provided with a rear stop, shall yield a punctual or stigmatic image of a plane object placed in the primary focal plane of the lens, is as follows:

$$\left(\frac{n}{n-1}F-S\right)^2+\frac{n+2}{n(n-1)^2}F_2^2-\frac{2n+1}{(n-1)^2}F_2F+\frac{2(n+1)}{n(n-1)}S.F_2=0;$$

where F denotes the refracting power of the lens, F_2 denotes the refracting power of the second surface, $1/S$ denotes the distance of the stop, and n denotes the index of refraction of the lens. If the stop is a rear stop at a distance of 30 mm. from the lens, and if $n=1.52$, show that the maximum value of the refracting power of a convex lens which will give a punctual image of a plane object placed in the primary focal plane is $F=+14.87$ dptr.

21. Using SEIDEL'S formulæ as given in § 193, show that the condition that an infinitely thin lens surrounded by air, and provided with a rear stop, shall give a punctual or stigmatic image of an infinitely distant object, is:

$$\left(\frac{n}{n-1}F+C\right)^2+\frac{n+2}{n}R_1^2-\frac{2n+1}{n-1}F.R_1-\frac{2(n+1)C}{n}R_1=0,$$

where n denotes the index of refraction of the lens, F denotes its refracting power, R_1 denotes the curvature of the first surface of the lens, and $C=S-F$, the magnitude S being equal to the reciprocal of the stop-distance.

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